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A One-dimensional Model of Thermo-electroelasticity in Extended Thermodynamics

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ABSTRACT: The subject of thermo-electroelasticity involves many complications due to the multiple ways in which the mechanical, thermal and electric fields can interact, some of these involving non-linearities. In extended thermodynamics, an additional difficulty arises due to the requirement of finiteness of the speed of propagation of the thermal disturbances. This implies, as may be observed in the extensive literature on the subject, a re-visiting of the basic postulates of thermodynamics, ultimately leading to the desired generalization. There are only a few nonlinear models dealing with this subject. In order to consider general nonlinear models, it is necessary to study linear ones first, as they represent most of the basic features of the studied phenomena. This is particularly true when the problem is tackled numerically through iteration methods, in which case the starting field equations are linear.

Here we study a one-dimensional system of equations of thermo-electroelasticity in extended thermodynamics and in the quasi-electrostatic regime. The nonlinear equations are given for reference only. The mixed character, parabolic-hyperbolic, of the associated linear system is established through the study of the characteristic curves. Two speeds of wave propagation are given in evidence, one for the usual coupled thermoelastic wave, and the other for a second sound. Parabolicity is due to the quasi-static distribution of the electric field. An example concerning the half-space is treated numerically by the Cranck-Nicolson method. The curves presented clearly illustrate the propagation of two types of waves, the usual coupled thermoelastic wave, and a faster wave generated by the second sound. It is hoped that the present study will clarify the basic features of the solution, as a prelude to tackling more sample, nonlinear equations.

Keywords: Electroelasticity; Extended thermodynamics; Linear theory; Characteristics; Cranck-Nicolson method.

نموذج أحادي البعد لنظرية المرونة الكهروحرارية في إطار الديناميكا الحرارية الموسعة

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الملخص: يتضمن موضوع المرونة الحرارية الكهربائية العديد من الصعوبات بسبب الطرق المتعددة للتفاعل بين المجالات: الميكانيكي، الحراري والكهربائي، وبعضها يتضمن ظواهر غير خطية. في الديناميكا الحرارية الموسعة، تنشأ صعوبة إضافية نظرا لمحدودية سرعة انتشار الاضطرابات الحرارية. وهذا يعني، كما يمكن أيضا ملاحظته في المرجعيات الواسعة النطاق حول هذا الموضوع، إعادة زيارة للمفاهيم الأساسية للديناميكا الحرارية، مما يؤدي في نهاية المطاف إلى التعميم المطلوب. يوجد عدد قليل فقط من النماذج غير الخطية التي تتناول هذا الموضوع، ومن أجل النظر في النماذج غير الخطية العامة، فمن الضروري دراسة النماذج الخطية أولا لأنها تمثل معظم السمات الأساسية للظواهر المدروسة. ويصدق ذلك بشكل خاص عندما تعالج المشكلة عدديا من خلال الطرق التكرارية، وفي هذه الحالة تكون معادلات خطوة البداية هى المعادلات الخطية.

وهنا ندرس نظاما أحادي البُعد لمعادلات نظرية المرونة الحرارية الكهربائية في الديناميكا الحرارية الموسعة وفي النظام شبه الكهروستاتيكي. وتعطَى المعادلات غير الخطية كمرجع فقط. ويتم دراسة النظام المختلط الناتج من المعادلات التفاضلية "مكافئ – ناقصي" من خلال النظام الخطي المرتبط بواسطة المنحنيات المميزة، وتوضع سرعتان في انتشار الموجة: واحدة للموجة المرنة الحرارية المقترنة والآخرى لما يُسمى بالصوت الثاني. ويرجع ذلك إلى التوزيع شبه الكهروستاتيكي للمجال الكهربائي، ويتم التعامل مع مثال يتعلق بنصف فراغ رقميا بواسطة طريقة كرانك - نيكلسن. وتوضح المنحنيات المعروضة بوضوح انتشار نوعين من الموجات: الموجة المرنة الحرارية المعتادة المقترنة، وموجة سريعة تم إنشاؤها بواسطة الصوت الثاني. ويؤمل أن توضح هذه الدراسة السمات الأساسية للحل تمهيدا لمعالجة المعادلات غير الخطية الأكثر وفرة.

الكلمات المفتاحية: المرونة الكهربائية - الديناميكا الحرارية الموسعة – النظرية الخطية – المنحنيات المميزة - طريقة كرانك - نيكلسن.

1. Introduction

The theory of thermo-electroelasticity in generalized thermodynamics may be of interest in the study of electrical disturbances which accompany the propagation of heat waves at low temperature (Rybalko *et al.* [1], Pashitskii and Ryabchenko [2], and Pashitskii *et al.* [3]). Dost [4] treats the case of thermoelastic dielectrics within the nonlinear theory. Ersoy [5, 6] considers electrically and thermally conducting magnetothermoelastic solids. His theory is based on the introduction of the electric current and the heat flux vectors as independent state variables. Montanaro [7] develops a model of nonlinear thermoelasticity in extended thermodynamics for electrically polarizable and finitely deformable, heat conducting elastic continua. Ghaleb [8] presents a fully nonlinear model for electrically polarizable, heat conducting elastic continuous media in the quasi-electrostatic approximation. Kuang [9] considers wave propagation in pyroelectrics and other materials with complex structure in extended thermodynamics within the linearized theory. Montanaro [10] treats the case of electrical continuous media within Green and Naghdi thermoelasticity theory. Chandrasekharaiah [11] develops a model for piezoelectrics with the heat flux as an independent state variable. Singh [12] investigates thermo-piezoelectric solids in extended thermodynamics using Green-Lindsay and Lord-Shulman theories. Zhou and Yang [13] investigate the propagation of plane waves in pyroelectric materials in the presence of viscous effects. Solutions of concrete problems are almost inexistant in the literature. Our attention was drawn lately to a recent publication concerning two-dimensional electro-magneto thermoelastic wave propagation in an electrically conducting cylinder, within the frame of extended thermodynamics with one relaxation time [14].

All of the above references illustrate a multitude of approaches, conceived to remove the paradox of infinite propagation of thermal disturbances and to incorporate the electric interactions. Most of them rely on rigorous thermodynamics, meaning that the basic laws of thermodynamics are respected, as well as the celebrated Clausius-Duhem inequality expressing the non-negativeness of the dissipation function. However, some of these approaches still need more solid experimental verification. A common ground to all of them is the need for an enrichment of the basic thermodynamical variables describing the electro-thermomechanical system. An interesting contribution discussing the difficulties facing the different models of extended thermodynamics may be found in [15].

For dielectric materials, which are the subject of the present investigation, we use the same approach as described in [7] and [8]. The basic set of thermodynamical parameters is enriched by adjoining the heat flow vector to the classical set involving strain and the electric field. That was shown to be consistent with the basic principles of thermodynamics in earlier work by Coleman *et al*. [16]. A new feature of the present work is the introduction of initial and boundary conditions for the heat flow vector, independently of those for temperature.

In what follows, we study a one-dimensional system of linear equations of thermo-electroelasticity in extended thermodynamics and in the quasi-electrostatic regime. Thus, any magnetic contributions are disregarded from the outset. The original nonlinear equations are given for reference only. It is hoped that the obtained results will be helpful in clarifying the basic features of the solution, thus laying the background for the numerical treatment the nonlinear equations using an iterative method. Two speeds of wave propagation are put in evidence, one for the usual coupled thermoelastic wave, and the other for second sound. A numerical example is treated for the half-space. All the unknown functions are zeroed at a sufficiently large distance from the boundary of the half-space, as a result of which reflected waves are expected to arise. In order to avoid them, the computational grid includes somewhat restricted time values. As noted above, future work is under progress to study nonlinear wave propagation in pyroelectric materials in two dimensions.

2. The nonlinear equations

The following equations are a restriction to one spatial dimension of a more general model of electro-thermoelasticity based on the introduction of the heat flow vector as an additional state variable in the free energy density of the medium and are introduced in [8]. The equations are in dimensionless form, the velocity of elastic waves being taken as unity:

$$(\gamma-1)U_x+\gamma_1 U_x^2+\nu E\Theta=-\varepsilon E-\beta_3\Theta, \tag{1}$$

$$U_{tt}-U_{xx}(1+2\gamma U_x+3\delta U_x^2)=-\alpha_1\Theta_x-\gamma_1 E_x+\beta_3(U_x\Theta)_x+f(x,t), \tag{2}$$

$$(\Theta+\beta_1 U_x+\beta_2 E-\frac{1}{2}bU_x^2)_t=-Q_x+\frac{1}{\Theta_0}Q\Theta_x+AQ^2+r(x,t), \tag{3}$$

$$(\Theta+\Theta_0)Q_t=-Q-\frac{1}{\kappa}\Theta_x. \tag{4}$$

Here, U denotes the mechanical displacement component, Θ is the absolute temperature as measured from a reference temperature Θ_0, Q is the heat flow vector and E is the electric field component. These four relations represent respectively: (i) the equation of electrostatics expressing the vanishing of the divergence of the electric induction, after integration and setting the integration constant equal to zero; (ii) the equation of motion; (iii) the equation of heat conduction and (iv) the Vernotte-Cattaneo law which replaces the usual Fourier law for heat conduction. The coefficient A in eq.(3) is the multiplicative factor of the squared heat flow vector in the expression for the free energy of the system. The other coefficients appearing in the equations denote material constants having obvious meaning. In particular, the coefficients γ and δ express the dependence of the elastic moduli on strain. Many of these coefficients will drop out in the following sections, when considerations are restricted to the linear equations. It may be noticed that the form of the Vernotte-Cattaneo law used in (4) and proposed in [8] does not yield the classical Fourier law for heat conduction in the limit of small relaxation times, and hence it cannot be used under such a restrictive condition.

Considering wave propagation in a half-space with initial uniform reference temperature Θ_0, the characteristic length L_0, time T_0 and heat flow vector Q_0 used to deduce the dimensionless equations are taken as follows:

$$L_0 = \frac{K}{\rho CV}, \quad T_0 = \frac{K}{\rho CV^2}, \quad Q_0 = \frac{\rho\Theta_0 CL_0}{T_0} = \rho\Theta_0 CV,$$

where ρ is the material density, C is the specific heat, K is the thermal conductivity and V is a characteristic velocity of propagation of the elastic waves. Tentatively, for common solids the different coefficients take on the following values in the SU system of units:

$$\Theta_0 = 10^3 K, \quad \rho = 10^4 kgm^{-3}, \quad C = 10^2 Jkg^{-1}K^{-1}, \quad K = 237 Wm^{-1}K^{-1}, \quad V = 10^5 ms^{-1}.$$

In what follows we consider a problem for the half-space $0 < x < \infty$. The formulation necessitates a boundary condition for the new variable of state Q, to be considered side by side with the boundary condition for temperature.

3. On the character of the system of linear equations

Following [8] and using a well-known formalism based on the introduction of the heat flow vector as an additional state variable, one gets the linear equations of electro-thermoelasticity in extended thermodynamics. In the one-dimensional case, after dimension analysis to reduce the speed of the classical coupled thermoelastic waves to unity, the governing linear equations may be written as a system of first-order partial differential equations in six unknown functions $\{U, \Theta, Q, E, P, R\}$ of the form:

$$\zeta\frac{\partial E}{\partial t} + \gamma\frac{\partial U}{\partial x} = R - \varepsilon E - \beta_3\Theta, \tag{5}$$

$$\frac{\partial P}{\partial t} - \frac{\partial P}{\partial x} + \alpha_1\frac{\partial \Theta}{\partial x} + \gamma\frac{\partial E}{\partial x} = f(x,t), \tag{6}$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial Q}{\partial x} + \beta_1\frac{\partial R}{\partial t} + \beta_2\frac{\partial E}{\partial t} = r(x,t), \tag{7}$$

$$\tau\frac{\partial Q}{\partial t} + \frac{\partial \Theta}{\partial x} = -Q, \tag{8}$$

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} = P, \tag{9}$$

$$\eta\frac{\partial R}{\partial t} + \frac{\partial U}{\partial x} = R, \tag{10}$$

The two unknown functions P and R are defined from the last two equations of the above system. The parameters ζ, η are two positive small parameters introduced artificially for convenience. Subsequently, they will be made to decrease during the computations, starting from some initial values. The function f represents the external forces of non-electric origin acting on the medium, while r denotes the volume heat supply. The other parameters appearing in the equations are dimensionless quantities involving the physical parameters of the medium.

Using standard analysis, it may be shown that all the characteristics of the considered linear system of equations are real, and do not depend on the two constants ζ, η introduced earlier. Four of these characteristics yield the speeds of wave propagation, for the usual coupled thermoelastic wave and for second sound:

$$v_{1,2} = \pm 1, \qquad v_{3,4} = \pm\frac{1}{\sqrt{\tau}}.$$

These are the same as for the purely thermoelastic case, showing that in this linear approximation, the electric field does not influence the velocities of wave propagation. The remaining two velocities are equal to zero.

Thus the equations under consideration form a system of partial differential equations of the first order of mixed, parabolic-hyperbolic type. It is believed that the parabolic element is due to the considered quasi-electrostatic approximation.

4. Numerical scheme

In what follows, we use the finite difference Crank-Nicholson method [17] to solve the initial set of linear equations (5)-(10). For the computational work, consider finite intervals on the x and t axes. The domain in the (x,t) plane is discretized by a grid with step length $\Delta x = h$ and time step $\Delta t = k.$

Let M, N be natural numbers, and the coordinates of the mesh points are:

$$x_n = nh,\ \ n = 0, 1, 2, ..., N, \qquad t_m = mk,\ \ m = 0, 1, 2, ..., M,$$

where

$$h = (b-a)/N, \qquad \Delta t = T/M, \qquad a \le x \le b, \qquad 0 \le t \le< T.$$

The numerical values of the variables E, U, P, θ, Q and R at the grid point (x_n, t_m) is denoted, respectively, by $E_n^m, U_n^m, P_n^m, \theta_n^m, Q_n^m$ and R_n^m. We use the following differences approximations:

1. For first order time derivative

$$(u_t)_n^m = \frac{u_n^{m+1} - u_n^m}{k} + O(k). \tag{11}$$

2. For first order space derivative

$$(u_x)_n^m = \frac{1}{2}\left(\frac{u_{n+1}^m - u_{n-1}^m}{2h} + \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1}}{2h}\right) + O(h)^2. \tag{12}$$

3. For source term

$$(f)_n^m = \frac{1}{2}(f_n^{m+1} + f_n^m). \tag{13}$$

These expressions will be used to approximate the partial derivatives of all the unknown functions E, U, P, θ, Q and R in the proposed system. The above replacement changes this system into a linear algebraic system for the unknowns $E_n^m, U_n^m, P_n^m, \theta_n^m, Q_n^m$ and R_n^m, and then the new system can be solved easily. Also, the local truncation error of this scheme is of the order $O(h^2 + k),$ and it is well known that the Crank-Nicholson scheme is unconditionally stable when used to solve hyperbolic partial differential equations of the first order [17].

The system of equations is solved under the following initial-boundary conditions:

$$E(x,0) = U(x,0) = P(x,0) = \theta(x,0) = Q(x,0) = R(x,0) = 0,$$

$$E(0,t) = 1 - \cos t, \qquad U(0,t) = 0.5(1 - \cos t), \qquad P(0,t) = p_0(1 - \cos t) + p_1 \sin t,$$

$$\theta(0,t) = 1 - \cos t, \qquad Q(0,t) = -Bi \cdot (\theta(0,t)), \qquad R(0,t) = r_0(1 - \cos t),$$

$$E(x_{final}, t) = U(x_{final}, t) = P(x_{final}, t) = \theta(x_{final}, t) = Q(x_{final}, t) = R(x_{final}, t) = 0,$$

and for the following values of the material and geometrical constants

$$f_1 = R - \varepsilon E - \beta_3 \theta,\ f_2 = 0.0012,\ \ f_3 = 0,\ f_4 = -Q.$$

The unknowns P and R are not independent, but defined through the other unknowns. Thus, the constants p_0, p_1 and r_0 cannot be arbitrary. Simple calculations show that

$$p_0 = \frac{r_0}{\gamma} - \frac{1}{\gamma}(\varepsilon + \beta_3), \quad p_1 = \frac{1}{2} - \frac{\zeta}{\gamma}, \quad r_0 = \frac{\varepsilon + \beta_3}{1 - \gamma}.$$

Taking

$$a = 0, b = 350, T = 10\pi, N = M = 35, \gamma = 0.1, \alpha_1 = 0.5,$$

$$\beta_1 = -0.1,\ \beta_2 = 0.5,\ \beta_3 = 0.5,\ \eta = \zeta = 0.0001,\ \tau = 0.1,\ \varepsilon = 0.98,\ Bi = 0.1.$$

All variables were zeroed at x_{final}, on the basis that waves have not reached this point at time t_{final} and therefore no reflected waves will appear during this period of time. The maximal value for time is t_{final} 35, while the values of the spatial coordinate x were allowed to run up to the value $x_{final} = 10 t_{final} \; 350$, in order to be able to view the fast wave travelling with velocity $\frac{1}{\sqrt{\tau}} \; 3.16$.

The solutions for the mechanical displacement, temperature, heat flow vector and electric field are represented as 3-D surfaces on Figures 1-4, in which one clearly notices two separate propagating waves generated by the boundary regimes. The computations were repeated many times for different values of η and ζ, and it was noted that all the figures remained unchanged as these two constants were made smaller by many orders of magnitude, thus justifying their introduction for convenience.

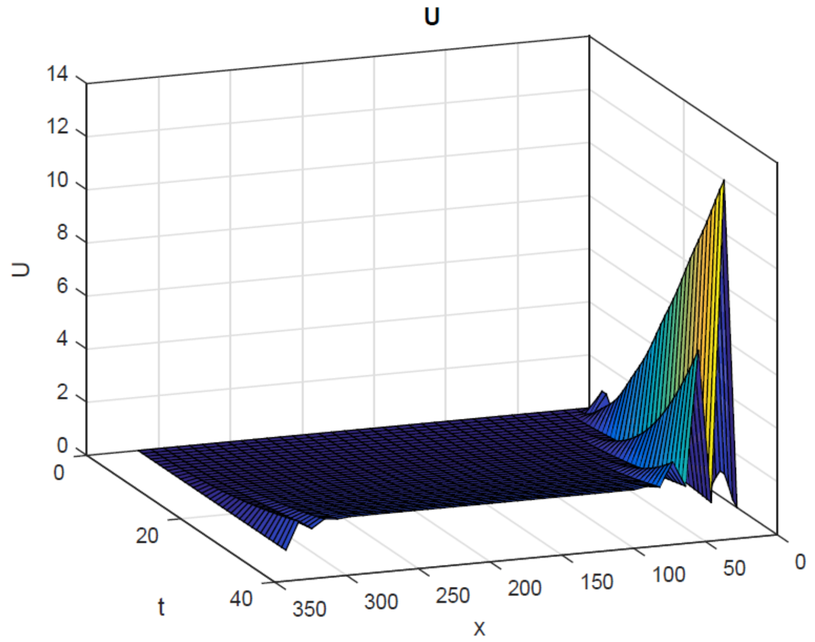

Figure 1. Mechanical displacement component U.

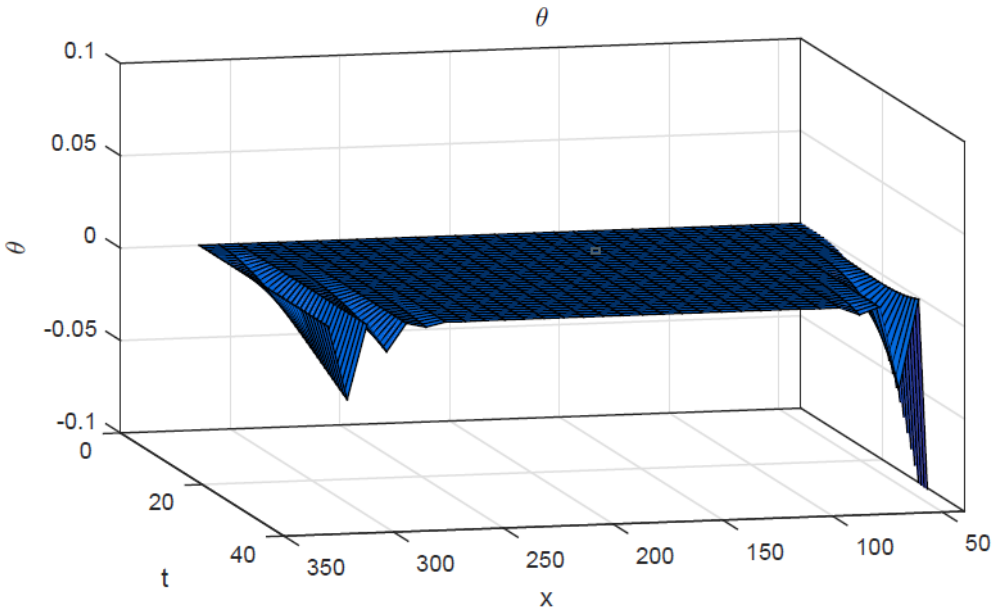

Figure 2. Temperature θ.

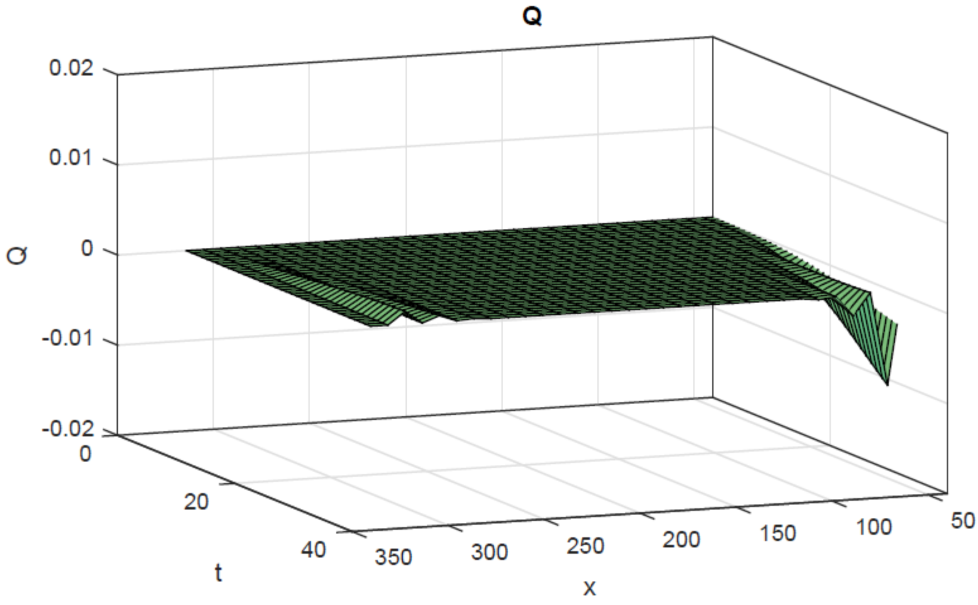

Figure 3. Heat flow vector component Q.

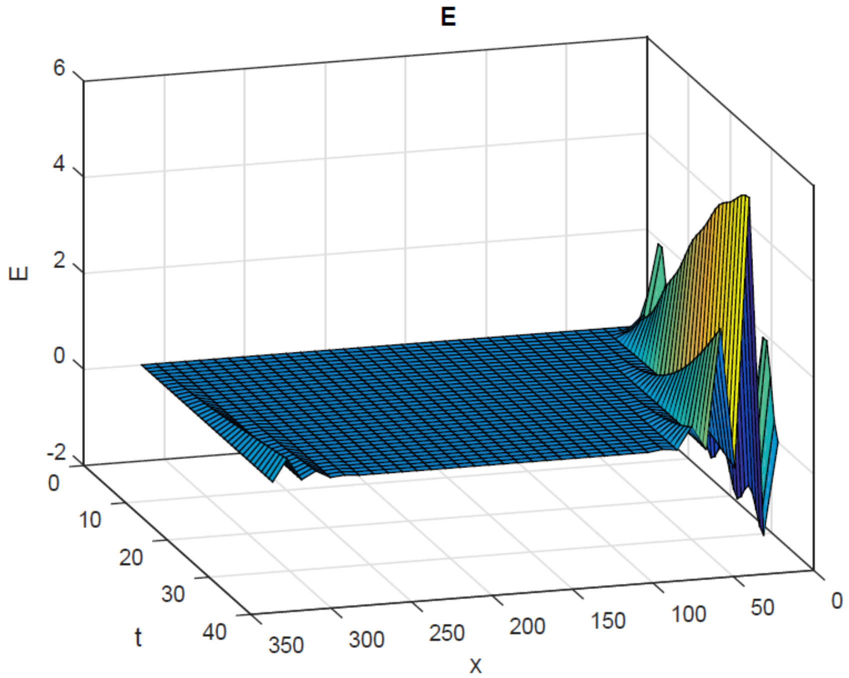

Figure 4. Electric field component E.

5. Conclusion

A model of linear thermo-electroelasticity in extended thermodynamics and in the quasi-electrostatic approximation has been investigated using the method of characteristics, and then numerically for given initial and boundary conditions. The main ingredient in this model is a Cattaneo-type evolution equation, which effectively requires the heat flow vector to be considered as an additional state function, independent of temperature. This requires the introduction of initial and boundary conditions for the heat flow vector, independently of those for temperature, a fact that represents a fundamental difference from the thermoelasticity problems solved within the frame of different theories of extended thermodynamics. For these, the additions of temporal derivatives occur in the heat equation itself. Moreover, the problems are usually solved by Laplace transform, with the elimination of the heat flow vector in favor of the other unknowns of the problem, hence not requiring additional limiting conditions for this variable. The presently used boundary condition for the heat flow vector is a Robin thermal condition which may be controlled experimentally. Other types of boundary conditions for the heat flow vector may equally well be used for the computations.

Dimension analysis and a study of the characteristic curves has revealed the mixed "parabolic-hyperbolic" character of the system of linear equations of electro-thermoelasticity. The parabolic element is due to the used quasi-electrostatic approximation. Moreover, within the linear theory, the electric field does not influence the speeds of propagation of the waves. This will certainly not be true anymore in the nonlinear case. A numerical application clearly shows two types of waves: The usual, coupled thermoelastic wave and the second sound, propagating in the medium.

It is hoped that the presented results, together with the future extension to include nonlinear interactions, will help investigating interesting problems in thermo-electroelasticity.

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