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Joint Symbol and Frame Synchronization for Direct- Detection Optical Communication Systems

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تزامن الرموز والإطارات في أنظمة الاتصالات الضوئية ذات الكشف المباشر

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خلاصة : يتعرض هذا البحث لمشكلة تزامن الرموز والإطارات في أنظمة الاتصالات الضوئية التي تستخدم الكشف المباشر, بافـتراض معـرفة توقيت الشقوق. ويستنبط البحث العلاقات الخاصة بتزامن الرموز والإطارات لحالات الاحتمالات القصوى المثلى وللحالات دون المثلى. كما يناقش البحث أسباب الخطأ في اشتقاق جورج هايدس.

ABSTRACT: The problem of joint symbol and frame synchronization in direct-detection optical PPM communication systems under the assumption of known slot timing is considered here. The optimum maximum-likelihood (ML) and sub-optimum rules for this joint symbol and frame synchronization problem are derived. The reason of Georghiades's (1985) incorrect ML rule is discussed in this paper.

KEYWORDS: Frame Synchronization, Direct-Detection, Optical PPM Communication, Optimum Maximum-Likelihood.

1. Introduction

Unfortunately, as will be discussed in this paper, Georghiades' derivation and his reported ML rule (Georghiades, 1985) is not correct. This is because that derivation did not consider the end effects of each frame properly and also invoked an invalid symmetry assumption to simplify the structure of the reported decision rule. In this paper, the correct optimum ML rule is derived. Simulation results presented here shows a significant improvement in correct synchronization probability performance of the correct optimum ML rule over Georghiades' incorrect ML rule. In particular, for high signal-to-noise ratios, the synchronization probability performance of the correct ML rule tends to the random data-limited upper bound while the performance of Georghiades' incorrect ML rule pre-saturates at a significantly lower level. We shall also consider a sub-optimum ML rule that accounts for the end effects of each frame, but also assumes the invalid symmetry assumption. This sub-optimum rule has a performance intermediate between that of the optimum ML rule and Georghiades' incorrect ML rule.

2. Joint Symbol and Frame Synchronization Problem

We consider PPM modulation over the direct-detection optical Poisson channel in which each M-ary symbol duration is divided into M time-slot divisions, and a rectangular light pulse is sent in the time slot associated with the transmitted symbol. The channel output is a Poisson process with intensity rate $\lambda_s + \lambda_n$ when a light pulse is transmitted and λ_n otherwise. Here λ_s is the photodetector count rate due to the light pulse and λ_n is the count rate due to dark current and background noise. Data transmission is formatted in successive frames with periodically inserted fixed synchronization patterns. Each frame is assumed to consist of N data symbols composed of a fixed L-symbol sync pattern and N-L random data symbols. No assumption is made to preclude the presence of the sync pattern among the random data symbols. We shall represent each M-ary symbol as a M-dimensional vector $\underline{d} = (d_0, ..., d_{M-1})$ where

$$d_i = \begin{cases} \lambda_s & ; \quad \text{if the light pulse is the i - th slot,} \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

Let the sync pattern be given by

$$\underline{S} = (\underline{S}_0, \underline{S}_1, ..., \underline{S}_{L-1}), \tag{1}$$

where for $0 \le i \le L-1$,

$$\underline{S}_i = (S_i^0, S_i^1, ..., S_i^{M-1}). \tag{2}$$

In the joint symbol and frame synchronization problem, the channel output corresponding to N transmitted symbols are observed. Since the pulse slot timing is known, the sufficient statistics are the photon counts in the NM time slots corresponding to the selected N transmitted symbols. Let

$$\underline{K} = (K_0, ..., K_{M-1}, K_M, ..., K_{2M-1}, ..., K_{(N-1)M}, ..., K_{NM-1}) \tag{3}$$

denote this vector of NM photon counts. There are NM possible starting positions for the sync pattern $\underline{S}$. The joint symbol and frame synchronization problem is to estimate this starting position. We consider the maximum likelihood approach here. The optimum ML rule estimates the sync pattern starting position as $\hat{m}$, where $0 \le \hat{m} \le NM-1$ is chosen to maximize the likelihood that $(K_{\hat{m}}, ..., K_{\hat{m}+LM-1})$ are the ML photon counts corresponding to the transmitted frame sync pattern $\underline{S}$.

2.1 ML Rule

Consider a candidate position m, $0 \le m \le NM-1$. The starting position of $\underline{S}$ corresponds then to the count K_m. So $(K_0, ..., K_{m-1})$ are the counts corresponding to random data symbols preceding $\underline{S}$, $(K_m, ..., K_{m+LM-1})$ are the counts corresponding to $\underline{S}$, and $(K_{m+LM}, ..., K_{NM-1})$ are the counts corresponding to random data symbols following $\underline{S}$. In order to consider all NM-candidate starting positions, we need to consider the (N-L) random data symbols preceding and following $\underline{S}$.

Hence denote

$$\underline{d} = (\underline{d}_L, ..., \underline{d}_{N-1}) \tag{4}$$

to be the N-L random data symbols following $\underline{S}$ and

$$\underline{\hat{d}} = (\underline{\hat{d}}_L, ..., \underline{\hat{d}}_{N-1}) \tag{5}$$

to be the N-L random data symbols preceding $\underline{S}$ where

$$\underline{d}_i = (d_i^0, ..., d_i^{M-1}), \tag{6}$$

$$\underline{\hat{d}}_i = (\hat{d}_i^0, ..., \hat{d}_i^{M-1}), \tag{7}$$

and

$$d_i^j, \hat{d}_i^j = \begin{cases} \lambda_s & ; \text{ if the pulse for the i}^{\text{th}}\text{symbol is in the j}^{\text{th}}\text{slot.} \\ 0 & ; \text{ otherwise} \end{cases} \tag{8}$$

Table 1 illustrates the relation between $\underline{K}, \underline{S}, \underline{d},$ and $\underline{\hat{d}}$ for N=4, L=2 and M=2 for each of the NM candidate starting positions m.

The ML rule chooses its estimate to be the value of m that maximizes

$$\Pr(\underline{K} \mid m) = \text{Probability distribution of } \underline{\text{K}} \text{ given m.} \tag{9}$$

Similar to the approach taken in [1], $\Pr(\underline{K} \mid m)$ can be derived by averaging over the random data $\underline{d}$ and $\underline{\hat{d}}$,

$$\Pr(\underline{K} \mid m, \underline{d}, \underline{\hat{d}}) = \text{Probability distribution of } \underline{\text{K}} \text{ given m}, \underline{\text{d}} \text{ and } \underline{\hat{\text{d}}}. \tag{10}$$

For the direct-detection optical channel the components of $\underline{K}$ are all conditionally independent Poisson random variables given m, $\underline{d}$ and $\underline{\hat{d}}$. In examining Table 1 it can be seen that there are three separate cases to consider

a) **<u>Case I</u>** $0 \le m \le (N-L)M$ and m (mod M) = 0

Suppose m=qM, where $0 \le q \le N-L$. Then $\Pr(\underline{K} \mid m, \underline{d}, \underline{\hat{d}})$ depends on (N-L-q) $\underline{d}_k$'s and q $\underline{\hat{d}}_k$'s for a total of (N-L) i.i.d. random data symbols.

b) **<u>Case II</u>** $(N-L)M+1 \le m \le NM-1$

$\Pr(\underline{K} \mid m, \underline{d}, \underline{\hat{d}}) = \Pr(\underline{K} \mid m, \underline{\hat{d}})$ depends on (N-L) i.i.d. $\underline{\hat{d}}_k$'s.

c) **<u>Case III</u>** $0 \le m \le (N-L)M-1$ and $m(\text{mod } M) = k \ne 0$

Table 1: Relationship between $\underline{S}, \underline{d}, \underline{\hat{d}}, \underline{K}$ for M=2,L=2,N=4.

m	m(mod M)	$K_0K_1K_2K_3K_4K_5K_6K_7$
0	0	$S_0^0S_0^1S_1^0S_1^1d_2^0d_2^1d_3^0d_3^1$
1	1	$\hat{d}_3^1S_0^0S_0^1S_1^0S_1^1d_2^0d_2^1d_3^0$
2	0	$\hat{d}_3^0\hat{d}_3^1S_0^0S_0^1S_1^0S_1^1d_2^0d_2^1$
3	1	$\hat{d}_2^1\hat{d}_3^0\hat{d}_3^1S_0^0S_0^1S_1^0S_1^1d_2^0$
4=(N-L)M	0	$\hat{d}_2^0\hat{d}_2^1\hat{d}_3^0\hat{d}_3^1S_0^0S_0^1S_1^0S_1^1$
5	1	$S_1^1\hat{d}_2^0\hat{d}_2^1\hat{d}_3^0\hat{d}_3^1S_0^0S_0^1S_1^0$
6	0	$S_1^0S_1^1\hat{d}_2^0\hat{d}_2^1\hat{d}_3^0\hat{d}_3^1S_0^0S_0^1$
7	1	$S_0^1S_1^0S_1^1\hat{d}_2^0\hat{d}_2^1\hat{d}_3^0\hat{d}_3^1S_0^0$

Here $\Pr(\underline{K} \mid m, \underline{d}, \underline{\hat{d}})$ is a function of N-L-q-1 entire $\underline{d}_k$ vectors and q entire $\underline{\hat{d}}_k$ vectors as well as a function of part of another $\underline{d}_k$ vector and part of another $\underline{\hat{d}}_k$ vector. The partial vectors are at the two ends of the NM-vector K. So $\Pr(\underline{K} \mid m, \underline{d}, \underline{\hat{d}})$ depends on N-L-q $\underline{d}_k$'s and q+1 $\underline{\hat{d}}_k$'s for a total of N-L+1 i.i.d. random data symbols. This case then differs significantly from the first two cases above.

Georghiades' incorrect derivation (Georghiades, 1985) of the ML rule makes the mistake of assuming that only cases I and II hold and does not consider case III. In order to derive $\Pr(\underline{K} \mid m)$, let us first consider $\Pr(\underline{K} \mid m, \underline{d}, \underline{\hat{d}})$ for each of the above three cases.

a) **Case I** $0 \le m \le (N-L)M$ and m (mod M) = 0

Assume that

$$m = qM, \tag{11}$$

where $0 \le q \le N-L$. Here

$$\Pr(\underline{K} \mid m, \underline{d}, \underline{\hat{d}}) = \prod_{i=0}^{L-1}\prod_{j=0}^{M-1} \frac{[(S_i^j + \lambda_n)T']^{K_{iM+j+m}}}{(K_{iM+j+m})!} e^{-(S_i^j+\lambda_n)T'}$$
$$* \prod_{i=L}^{N-1-q}\prod_{j=0}^{M-1} \frac{[(d_i^j + \lambda_n)T']^{K_{iM+j+m}}}{(K_{iM+j+m})!} e^{-(d_i^j+\lambda_n)T'} * \prod_{i=N-q}^{N-1}\prod_{j=0}^{M-1} \frac{[(\hat{d}_i^j + \lambda_n)T']^{K_{iM+j+m}}}{(K_{iM+j+m})!} e^{-(\hat{d}_i^j+\lambda_n)T'} \tag{12}$$

where T' is the pulse slot duration. We also adopt the convention here $\prod_{i=m}^{n} f(i) = 1$ whenever n<m. Moreover indices are interpreted modulo NM.

b) **Case II** $(N-L)M + 1 \le m \le NM - 1$

Here

$$\Pr(\underline{K} \mid m, \underline{d}, \underline{\hat{d}}) = \Pr(\underline{K} \mid m, \underline{\hat{d}})$$
$$= \prod_{i=0}^{L-1}\prod_{j=0}^{M-1} \frac{[(S_i^j + \lambda_n)T']^{K_{iM+j+m}}}{(K_{iM+j+m})!} e^{-(S_i^j+\lambda_n)T'} * \prod_{i=L}^{N-1}\prod_{j=0}^{M-1} \frac{[(\hat{d}_i^j + \lambda_n)T']^{K_{iM+j+m}}}{(K_{iM+j+m})!} e^{-(\hat{d}_i^j+\lambda_n)T'}. \tag{13}$$

c) **Case III** $0 \le m \le (N-L)M - 1$ and $m (\mathrm{mod}\, M) = k \ne 0$

Assume that

$$m = qM + k, \tag{14}$$

where $0 \le q \le N-L-1$ and $0 < k \le M-1$.

Here

$$\Pr(\underline{K} \mid m, \underline{d}, \underline{\hat{d}}) = \prod_{i=0}^{L-1}\prod_{j=0}^{M-1} \frac{[(S_i^j + \lambda_n)T']^{K_{iM+j+m}}}{(K_{iM+j+m})!} e^{-(S_i^j+\lambda_n)T'} * \prod_{i=L}^{(N-1-q)-1}\prod_{j=0}^{M-1} \frac{[(d_i^j + \lambda_n)T']^{K_{iM+j+m}}}{(K_{iM+j+m})!} e^{-(d_i^j+\lambda_n)T'}$$
$$* \prod_{j=0}^{M-1-k} \frac{[(d_{N-1-q}^j + \lambda_n)T']^{K_{(N-1)M+k+j}}}{(K_{(N-1)M+k+j})!} e^{-(d_{N-1-q}^j+\lambda_n)T'} * \prod_{j=M-k}^{M-1} \frac{[(\hat{d}_{N-1-q}^j + \lambda_n)T']^{K_{(N-1)M+k+j}}}{(K_{(N-1)M+k+j})!} e^{-(\hat{d}_{N-1-q}^j+\lambda_n)T'}$$
$$* \prod_{i=N-q}^{N-1}\prod_{j=0}^{M-1} \frac{[(\hat{d}_i^j + \lambda_n)T']^{K_{iM+j+m}}}{(K_{iM+j+m})!} e^{-(\hat{d}_i^j+\lambda_n)T'} \tag{15}$$

Note that

$$\prod_{i=0}^{L-1}\prod_{j=0}^{M-1} e^{-L(\lambda_s + M\lambda_n)T'} \tag{16}$$

and that

$$\prod_{i=0}^{N-1}\prod_{j=0}^{M-1} \frac{(\lambda_n T')^{K_{iM+j+m}}}{K_{iM+j+m}!} \tag{17}$$

are independent of m. Also note for the following cases that

a) **Case I**

$$\prod_{i=L}^{N-1-q}\left(\prod_{j=0}^{M-1} e^{-(d_i^j+\lambda_n)T'}\right)\prod_{i=N-q}^{N-1}\left(\prod_{j=0}^{M-1} e^{-(\hat{d}_i^j+\lambda_n)T'}\right) = e^{-(N-L)(\lambda_s+M\lambda_n)T'}, \tag{18}$$

b) **Case II**

$$\prod_{i=L}^{N-1}\prod_{j=0}^{M-1} e^{-(\hat{d}_i^j+\lambda_n)T'} = e^{-(N-L)(\lambda_s+M\lambda_n)T'}, \tag{19}$$

c) **Case III**

$$\prod_{i=L}^{(N-1-q)-1}\left(\prod_{j=0}^{M-1} e^{-(d_i^j+\lambda_n)T'}\right)\prod_{i=N-q}^{N-1}\left(\prod_{j=0}^{M-1} e^{-(\hat{d}_i^j+\lambda_n)T'}\right) = e^{-(N-L-1)(\lambda_s+M\lambda_n)T'}, \tag{20}$$

Define

$$C_1 = e^{-N(\lambda_s+M\lambda_n)T'}\prod_{i=0}^{N-1}\prod_{j=0}^{M-1}\frac{(\lambda_n T')^{K_{iM+j+m}}}{K_{iM+j+m}!} \tag{21}$$

(C_1 is independent of m). Then using (16) - (20) in (12), (13) and (15) we obtain for the three cases the following expressions for $\Pr(\underline{K}\mid m,\underline{d},\underline{\hat{d}})$.

a) **Case I** $m = qM,\ 0 \le q \le N-L$.

$$\Pr(\underline{K}\mid m,\underline{d},\underline{\hat{d}}) = C_1\prod_{i=0}^{L-1}\prod_{j=0}^{M-1}(1+S_i^j/\lambda_n)^{K_{iM+j+m}} * \prod_{i=L}^{N-1-q}\prod_{j=0}^{M-1}(1+d_i^j/\lambda_n)^{K_{iM+j+m}} * \prod_{i=N-q}^{N-1}\prod_{j=0}^{M-1}(1+\hat{d}_i^j/\lambda_n)^{K_{iM+j+m}} \tag{22}$$

b) **Case II** $(N-L)M+1 \le m \le NM-1$.

$$\Pr(\underline{K}\mid m,\underline{d},\underline{\hat{d}}) = C_1\prod_{i=0}^{L-1}\prod_{j=0}^{M-1}(1+S_i^j/\lambda_n)^{K_{iM+j+m}} * \prod_{i=L}^{N-1}\prod_{j=0}^{M-1}(1+\hat{d}_i^j/\lambda_n)^{K_{iM+j+m}} \tag{23}$$

c) **Case III** $m = qM+k,\ 0 \le q \le N-L-1,\ 0 < k \le M-1$.

$$\Pr(\underline{K}\mid m,\underline{d},\underline{\hat{d}}) = e^{\lambda_s T'}C_1\prod_{i=0}^{L-1}\prod_{j=0}^{M-1}(1+S_i^j/\lambda_n)^{K_{iM+j+m}} * \prod_{i=L}^{N-1-q}\prod_{j=0}^{M-1}(1+d_i^j/\lambda_n)^{K_{iM+j+m}} * \prod_{i=N-q}^{N-1}\prod_{j=0}^{M-1}(1+\hat{d}_i^j/\lambda_n)^{K_{iM+j+m}}$$

$$* \prod_{j=0}^{M-1-k} e^{-d_{N-1-q}T'}(1+d_{N-1-q}^j/\lambda_n)^{K_{(N-1)M+k+j}} * \prod_{j=M-k}^{M-1} e^{-\hat{d}_{N-1-q}T'}(1+\hat{d}_{N-1-q}^j/\lambda_n)^{K_{(N-1)M+k+j}}. \tag{24}$$

We next use (22) - (24) to obtain

$$\Pr(\underline{K}\mid m) = \sum_{\underline{d}}\sum_{\underline{\hat{d}}} M^{-2(N-L)}\Pr(\underline{K}\mid m,\underline{d},\underline{\hat{d}}). \tag{25}$$

In order to do this, define

$$C_2 = C_1 M^{-(N-L)}, \tag{26}$$

which is independent of m. Also define for each i, $0 \le i \le L-1$,

$$\hat{j}(i) = \text{pulse slot position for the i-th symbol of the sync pattern } \underline{S}, \tag{27}$$

and

$$x = (1+\lambda_s/\lambda_n). \tag{28}$$

It then follows that

$$\prod_{j=0}^{M-1}(1+S_i^j/\lambda_n)^{K_{iM+j+m}} = x^{K_{iM+\hat{j}(i)+m}}, \tag{29}$$

$$\sum_{\underline{d}_i}\prod_{j=0}^{M-1}(1+d_i^j/\lambda_n)^{K_{iM+j+m}} = \sum_{j=0}^{M-1}x^{K_{iM+j+m}}, \tag{30}$$

$$\sum_{\underline{d}_{N-1-q}}\prod_{j=0}^{M-1-k}e^{-d_{N-1-q}^jT'}(1+d_{N-1-q}^j/\lambda_n)^{K_{(N-1)M+k+j}} = e^{-\lambda_s T'}\sum_{j=0}^{M-1-k}x^{K_{(N-1)M+k+j}} + k \tag{31}$$

and

$$\sum_{\hat{\underline{d}}_{N-1-q}}\prod_{j=M-k}^{M-1}e^{-\hat{d}_{N-1-q}T'}(1+\hat{d}_{N-1-q}/\lambda_n)^{K_{(N-1)M+k+j}} = e^{-\lambda_s T'}\sum_{j=M-k}^{M-1}x^{K_{(N-1)M+k+j}} + (M-k). \tag{32}$$

Using (29) - (32) in (25) then yields a relatively simpler expression for $\Pr(\underline{K}\mid m)$. We express the results for the three above cases below in terms of

$$L(m) = \ln\Pr(\underline{K}\mid m) - \ln C_2 \tag{33}$$

For the ML rule, it is equivalent to choose m to maximize L(m) since C_2 is independent of m. The expressions for L(m) are identical in cases I and II, but different in case III.

a) **Case I and Case II** Either m(mod M) = 0 or $(N-L)M \le m \le NM-1$

$$L(m) = \ln(x)\sum_{i=0}^{L-1}K_{iM+\hat{j}(i)+m} + \sum_{i=L}^{N-1}\ln(\sum_{j=o}^{M-1}x^{K_{iM+j+m}}). \tag{34}$$

b) **Case III** $m(\text{mod M}) = k \ne 0$ and $0 \le m \le (N-L)M-1$.

$$L(m) = \ln(x)\sum_{i=0}^{L-1}K_{iM+\hat{j}(i)+m} + \sum_{i=L}^{N-1}\ln\left(\sum_{j=0}^{M-1}x^{K_{iM+j+m}}\right) - \ln\left(\sum_{j=0}^{M-1}x^{K_{(N-1)M+k+j}}\right) + \ln\left[e^{-\lambda_s T'}\left(\sum_{j=0}^{M-1-k}x^{K_{(N-1)M+k+j}}\right)+k\right]$$
$$+\ln\left[e^{-\lambda_s T'}\left(\sum_{j=M=k}^{M-1}x^{K_{(N-1)M+k+j}}\right)+(M-k)\right]+\lambda_s T' - \ln(M) \tag{35}$$

The ML rule is to choose m, $0 \le m \le NM-1$ to maximize L(m), where L(m) is given by (34) for cases I and II, and by (35) for case III. The expression for L(m) differs in case III from cases I and II because of the two partial symbols at the ends of the frame. Note from (34) and (35) that computation of L(m) for each m uses the counts corresponding to all N symbols in the frame. Georghiades' incorrect ML rule is based on a likelihood formula which for each m uses only the counts corresponding to the L symbols of the sync pattern. Since a low sync pattern overhead is desired for high data rate throughputs, the frame length N is often significantly larger than L in practice. Moreover, reasonably large values of L are usually employed to reduce the probability of replication of the sync pattern in the random data portion of the frame. Hence it is meaningful to consider the case when N becomes large relative to both L and M. In particular, the complexity of implementing the decision rule should be examined. Table 2 gives the total number of computations in terms of the number of addition, multiplications, integer power, logarithm and exponential function evaluations to compute L(m) for all m.

In this table, there are a total of N+L(M-1) m's for cases I and II and N(M-1)-L(M-1) m's for case III. The total number of computations is evaluated in cases I and II for all N+L(M-1) m's and also in case III for all N(M-1)-L(M-1) m's.

It can be seen from this table that, asymptotically for large N and fixed L and M, the complexity of implementing the ML rule based on computing L(m) is of order N^2. Georghiades' incorrect ML instead has an asymptotic complexity of order N. In order to obtain this reduced complexity Georghiades makes the incorrect assumption that A(m) is independent of m for all $0 \le m \le NM-1$, where

$$A(m) = \sum_{i=0}^{N-1} \ln\left(\sum_{j=0}^{M-1} x^{K_{iM+j+m}} \right). \tag{36}$$

Table 2: Total number of computations for ML rule.

Total Number of Computations in Cases I and II	
Computation	Total Number
Addition	[NM-LM+L-1][N+L(M-1)]
Multiplication	N+L(M-1)
Integer Power	[NM-LM][N+L(M-1)]
Logarithm	[N-L+1][N+L(M-1)]
Total Number of Computations in Case III	
Computation	Total Number
Addition	[(N-L)M+L+5][N(M-1)-L(M-1)]
Multiplication	3[N(M-1)-L(M-1)]
Integer Power	[NM-LM][N(M-1)-L(M-1)]
Logarithm	[N-L+3][N(M-1-(L(M-1)]
Exponential	2[N(M-1)-L(M-1)]

Suppose this assumption is true. Then the ML rule can be based on choosing m to maximize L(m) - A(m). The expressions for L(m)-A(m) in cases I, II and III are as follows.

a) **<u>Case I and Case II</u>** Either m(mod M) = 0 or $(N-L)M \le m \le NM-1$.

$$L(m) - A(m) = \ln(x)\sum_{i=0}^{L-1} K_{iM+\hat{j}(i)+m} - \sum_{i=0}^{L-1} \ln\left(\sum_{j=0}^{M-1} x^{K_{iM+j+m}} \right). \tag{37}$$

b) **<u>Case III</u>** $\text{m(mod M)} = k \ne 0$ and $0 \le m \le (N-L)M-1$.

$$L(m) - A(m) = \ln(x)\sum_{i=0}^{L-1} K_{iM+\hat{j}(i)+m} - \sum_{i=0}^{L-1} \ln\left(\sum_{j=0}^{M-1} x^{K_{iM+j+m}} \right) - \ln\left(\sum_{j=0}^{M-1} x^{K_{(N-1)M+k+j}} \right) + \ln\left[e^{-\lambda_s T'} \left(\sum_{j=0}^{M-1-k} x^{K_{(N-1)M+k+j}} \right) + k \right]$$
$$+ \ln\left[e^{-\lambda_s T'} \left(\sum_{j=M-k}^{M-1} x^{K_{(N-1)M+k+j}} \right) + (M-k) \right] + \lambda_s T' - \ln(M) \tag{38}$$

The asymptotic computational complexity of computing all of the L(m)-A(m) is of order N. Unfortunately, A(m) is not independent of m for all $0 \le m \le NM-1$. Hence the ML rule cannot be implemented by choosing m directly to maximize L(m)-A(m).

2.2 Simplified ML Rule

Although A(m) is not independent of m for all $0 \le m \le NM-1$, it is independent of m for restricted sets of integers m. Specifically, define for each k, $0 \le k \le M-1$,

$$B_k = \{m : 0 \le m \le NM-1 \text{ and } m(\text{mod } M) = k\}. \tag{39}$$

It can then be seen from (36) that for each k, $0 \le k \le M-1$, A(m) is independent of m for all $m \in B_k$. This property suggests the following approach towards reducing the number of computations to implement the ML rule. In order to describe this approach define

$$\max_{x \in A}{}^{-1}\{f(x)\} \tag{40}$$

to be the value of x achieving the following maximum:

$$\max_{x \in A}\{f(x)\} \tag{41}$$

First note that since A(m) is independent of m for $m \in B_k$,

$$m_k = \max_{m \in B_k}{}^{-1}\{L(m)\} \quad = \max_{m \in B_k}{}^{-1}\{L(m) - A(m)\} \tag{42}$$

But

$$\max_{0 \le m \le NM-1}{}^{-1}\{L(m)\} = \max_{0 \le k \le M-1}{}^{-1}\left\{\max_{m \in B_k}\{L(m)\}\right\} \quad = \max_{0 \le k \le M-1}{}^{-1}\{L(m_k)\} \tag{43}$$

Equations (42) and (43) suggest the following two-step implementation of the ML rule.

Step 1

For each k, $0 \le k \le M-1$, determine

$$\mathrm{m}_\mathrm{k} = \max_{m \in B_k}{}^{-1}\{L(m) - A(m)\}. \tag{44}$$

Step 2

The ML decision rule is the value $\hat{m}$ given by

$$\hat{m} = \max_{0 \le k \le M-1}{}^{-1}\{L(m_k)\}. \tag{45}$$

The advantage of this two step approach is because the asymptotic complexity (fixed M, L asymptotic in N) of computing the [L(m)-A(m)]'s for all $m \in B_k$ is of order N for each k. So the asymptotic complexity of step 1 is of order N. The asymptotic complexity of computing each of the $L(m_k)$'s in step 2 is also of order N. So the total number of computations involved in implementing steps 1 and 2 is asymptotically of order N. Table 3 gives the exact number of additions multiplications, integer powers, exponential and logarithm function evaluations. We shall call this rule the simplified ML rule.

Table 3: Total number of computations for simplified ML rule.

Total Number of Computations for Step 1	
Computation	Total Number
Addition	(N-L)(M-1)[M(L+2)+L+4]+[N+(M-1)L](ML+L-1)
Multiplication	[3(N-L)+L](M-1)+N
Integer Power	(N-L)(M-1)M(L+2)+[N+L(M-1)]ML
Logarithm	(N-L)(M-1)(L+5)+[N+(M-1)L](L+1)
Exponential	2(N-L)(M-1)
Total Number of Computations for Step 2	
Computation	Total Number
Addition	[(N-L)M+L+5](M-1)+(N-L)M+L-1
Multiplication	3M-2
Integer Power	[(N-L)M](M-1)+(N-L)M
Logarithm	(N-L+3)(M-1)+N-L+1
Exponential	2 (M-1)

2.3 Sub-optimum Rule

The incorrect Georghiades ML rule is invalid because it does not consider the case III expression for L(m) and also because it assumes the constancy of A(m) for all m. We next consider sub-optimum rule which assumes just the constancy of A(m). This rule then chooses m to achieve the following maximum,

$$\max_{0 \le m \le NM-1} \{L(m) - A(m)\}. \tag{46}$$

Similar to the above discussion on the simplified ML rule, the asymptotic complexity of this sub-optimum rule is of order N. It is interesting to consider this rule because it only assumes the constancy of A(m) for all m and does not also ignore the case III expression for L(m) as does Georghiades' incorrect ML rule. Hence, in comparing the performance between the ML rule, the sub-optimum rule and Georghiades' incorrect ML rule, it is possible to access the performance deteriorations due to these two incorrect assumptions. The section below discusses the relevant performance results.

3. Simulation Results

Computer simulation was used to evaluate the correct synchronization probability performance. Figures 1 to 6 give the simulation results on the correct synchronization probability of the ML, sub-optimal and incorrect ML rules for (N,L,M)=(4,2,2), (8,2,4), (12,2,8), (12,4,8), (12,6,8) and (20,4,4) respectively. These Figures give the correct synchronization probability as a function of the average signal count $\lambda_s T'$ per slot for an average noise count per slot $\lambda_n T' = 4$ photons. It can be seen that the true ML rule performs from 3 to 5 db better than the incorrect ML rule. The performance of the sub-optimal rule is intermediate between that of the ML rule and the incorrect ML rule. In fact the performance of the sub-optimal rule is closer to that of the ML rule than to that of the incorrect ML rule. This suggests that the constancy of the A(m) is a more robust assumption than ignoring the case III expression for L(m).

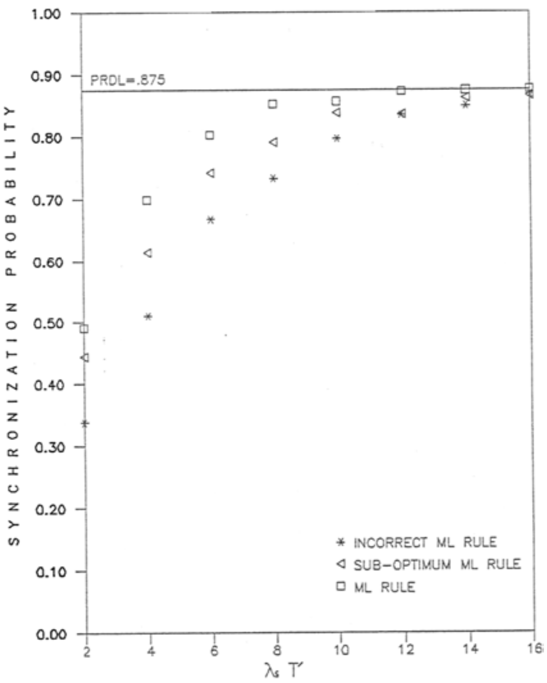

FIGURE 1 : PPM,NOISE INTENSITY=4
N=4, L=2, M=2

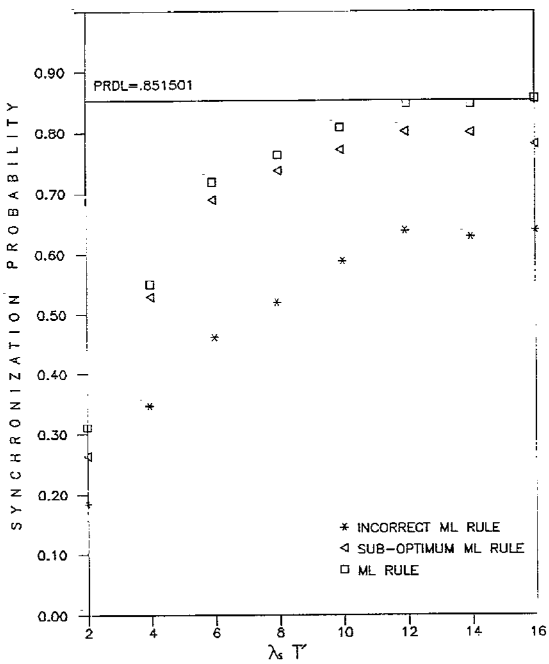

FIGURE 2 : PPM,NOISE INTENSITY=4
N=8, L=2, M=4

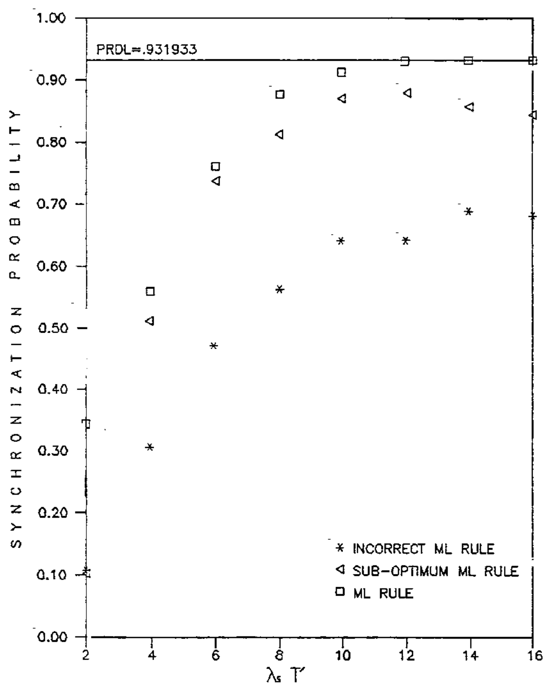

FIGURE 3 : PPM,NOISE INTENSITY=4
N=12, L=2, M=8

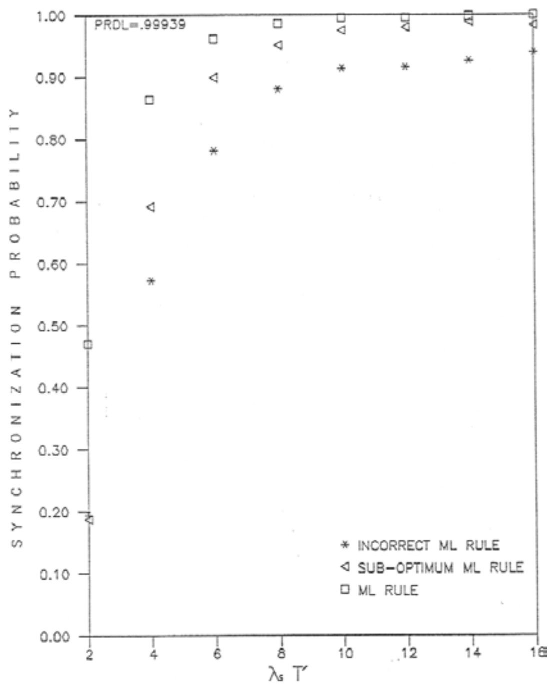

FIGURE 4 : PPM,NOISE INTENSITY=4
N=12, L=4, M=8

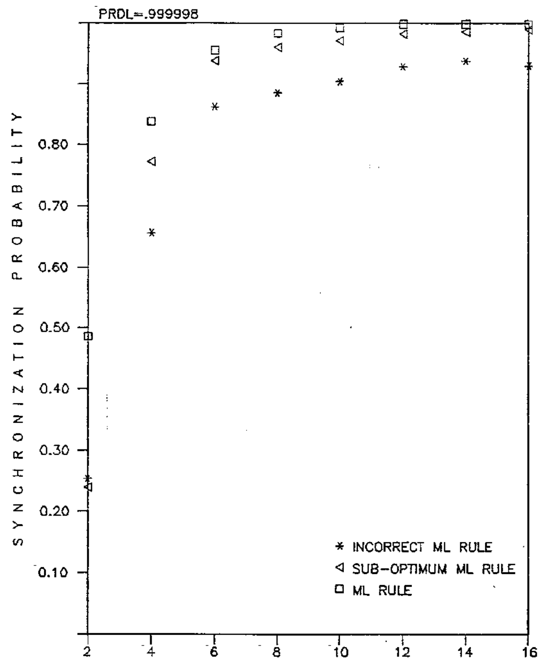

FIGURE 5 : PPM,NOISE INTENSITY=4
N=12, L=6, M=8

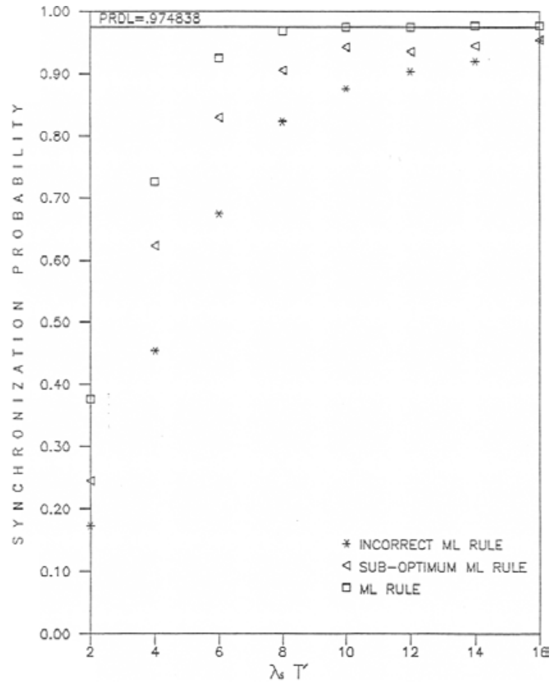

FIGURE 6 : PPM,NOISE INTENSITY=4
N=20, L=4, M=4

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