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Deletion Designs

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التصاميم الإنشاطية

مايكل كامايو جاشي ، جون ويكلاف أوديامبو و آن واشيرا

خلاصة : يقدم هذا البحث طريقة لإنشاء نوع من التصاميم العاملية ذات التكرارات المحددة المرنة في شكل مجموعات ويعطي تعابير جبرية لإيجاد نقص المعلومات في التفاعلات ذات الرتب المنخفضة.

ABSTRACT: In this paper a method of constructing a class of flexible single replicate factorial designs in blocks is given. Simple expressions for calculating loss of information on low order interactions is presented.

KEYWORDS: Asymmetrical Factorial Designs, Generalised Cyclic Designs, Loss of Information.

1. Introduction

Consider a single replicate factorial experiment involving n factors $F_1, F_2, ..., F_n$; factors F_i occurring at s_i levels. Let $\boldsymbol{a} = a_1 a_2 ... a_n$ denote a treatment combination, where a_i ($0 \le a_i \le s_i - 1$) is the level of factor F_i. The number of treatment combinations is then given by

$$v = \prod_{i=1}^{n} s_i \tag{1.1}$$

These v treatment combinations will always be *lexicographically ordered.* That is a treatment combination $\boldsymbol{a} = a_1 a_2 ... a_n$ appears before another treatment combination $\boldsymbol{a} = a_1^* a_2^* ... a_n^*$ if and only if for the first u such that $a_u \ne a_u^*$ we have $\boldsymbol{a_u} < \boldsymbol{a_u^*}$ for $1 \le u \le n$.

Suppose we wish to construct a $v = \prod_{i=1}^{n} s_i$ single replicate factorial experiment in blocks. We first construct a single replicate $r_1 \times r_2 \times ... \times r_n$ single replicate preliminary block design, say d_p, using one of the known methods, such that $r_i \ge s_i$ for $i = 1, 2, ..., n$. We can then select $l_i = r_i - s_i$, levels of the *i-th* factor of d_p and delete from d_p all treatment combinations where factor F_i occurs at any of the l_i selected levels. The resulting $s_1 \times s_2 \times ... \times s_n$ single replicate design is referred to as a *j-th* order deletion design if levels are deleted from j factors.

Bose (1947) laid the foundation of factorial designs. He used finite Euclidean geometry to construct symmetrical factorial designs in blocks. Kishen and Srivastava (1959) extended the method of finite geometries to the construction of balanced confounded asymmetrical factorial designs thereby introducing the idea of deletion. John and Dean (1975) proposed a simple method of confounding based on the properties of generalised cyclic designs from a set of generating treatments or generators and showed that the confounding patterns could easily be determined from these generators. More recently Voss (1986) has constructed nearly orthogonal single replicate factorial designs in blocks. He uses the deletion technique where he deletes from the first factor, without loss of generality, to obtain first order deletion designs. The most recent contribution in this direction is that of Chauhan (1989) who generalized the work by Voss (1986), by constructing efficient single replicate designs using the generalized deletion technique. Starting from an s^n single replicate generalized cyclic design, levels are deleted from the first m_1 factors, without loss of generality , to obtain an $(s-l)^{m_1} s^{n-m_1}$ deletion design.

The overall objective of the present study is to give results for the general order deletion designs of the form $s^{n-m_1}(s-l)^{m_1}$ which are proper, for $1 \le l \le s-1$ and m_1 less than or equal to the number of generators of the preliminary single replicate generalized cyclic design. The efficient proper single replicate designs of the form $(s-l)s^{n-1}$ given by Chauhan (1989) thus become special cases of the results obtained in this study. The method proposed by John and Dean (1975) is used to construct the preliminary single replicate factorial design, which is always symmetric. That is, factor F_i occurs at $s_i = s$ levels for all $i = 1,2,...,n$. Conditions are given which guarantee the existence of either proper or improper deletion designs. Simple formulas for calculating the loss of information, due to confounding with blocks, on main effects and two factor interactions are given. A simple method of choosing a fraction for estimating main effects and low order interactions is also given.

2. Notations

We shall first assume the fixed effects linear model

$$y_{ah} = \mu + \tau_a + \beta_h + \varepsilon_{ah} \tag{2.1}$$

where y_{ah} denotes the observed yield from treatment combination ***a*** in the *h-th* block; τ_a denotes the effect of the treatment combination ***a*** ; β_h denotes the effect of the *h-th* block and ε_{ah} are uncorrelated random errors with mean zero and variance σ^2. Let $y = (y_{ah})$ and $\tau = (\tau_a)$ denote $v \times 1$ vectors of observations and treatment effects respectively, each lexicographically ordered by ***a*.** That is the *i-th* row corresponds to the *i-th* treatment in the above arrangement of the *v* treatment combinations. We shall denote the incidence matrix, the intrablock matrix, the diagonal matrix of block sizes and the number of blocks, respectively, by ***N, A, K*** and b . The *i-th* row of the incidence matrix ***N*** corresponds to the *i-th* lexicographically ordered treatment combination a . The $q \times 1$ vectors of ones and of zeros will be denoted by 1_q and 0_q, respectively. A generalized interaction will be denoted by a^x where $x = x_1 x_2 ... x_n$ such that $x_j = 1$ if F_j is present in the interaction and $x_j = 0$ otherwise. A $v \times 1$ contrast vector will be denoted by c^x where

$$c^x = c_1^{x_1} \otimes c_2^{x_2} \otimes .... \otimes c_n^{x_n} \tag{2.2}$$

with $c_i^{x_i}$ being an $s_i \times 1$ vector of ones if $x_j = 0$, otherwise it is an $s_i \times 1$ contrast vector. The minimum variance unbiased estimator of the generalized interaction a^x is represented by $c^{x'}\tilde{\tau}$, where $\tilde{\tau}_i$, the *i-th* element of $\tilde{\tau}$, is the estimate of the fixed effect of the *i-th* treatment combination.

We shall denote the set of *n* factors $\{F_1, F_2, ..., F_n\}$ by $\{1,2,...,n\}$. Then for a non-empty subset $\{i_1, i_2, ..., i_r\} \subset \{1,2,..,n\}$, $V(i_1, i_2, ..., i_r)$ will denote the vector (factorial) space of contrast vectors

c^x corresponding to the estimator of the generalized interaction α^x, where $\{j\} \subset \{i_1, i_2, ..., i_r\}$ and $x_j = 0$ otherwise. All the notations corresponding to the preliminary design will carry the subscript p while those corresponding to the deletion design will carry no subscript.

3. Some Properties of Deletion Designs

We start by giving results useful in constructing deletion designs which can be used to estimate the main effects and also the results are useful in calculating loss of information due to confounding in blocks.

Theorem 1: If S^n is a generalized cyclic design generated by m generators such that g_1= sum of the first $(n - m + 1)$ rows of an identity matrix of order n and $g_2, g_3, ..., g_m$ are the last $(m - 1)$ rows of an identity matrix of order n, then there exists a proper deletion design of order m_1, provided $m_1 \le m$.

Let D_j be the $r_i \times s_j$ matrix obtained from an $s_j \times s_j$ identity matrix by deleting the *t-th* row if the *t-th* level is deleted from factor F_j in the preliminary design d_p to obtain the deletion design d. In our study, $s_j = s, j = 1, 2, ..., n$. We now state the following result.

Lemma 1: Suppose levels are deleted in descending order from factor F_j and let P^{a_j} be an $s \times s$ permutation matrix with 1 in the a_j *-th* column of the 0-*th* row. Then for $a_j \le s/2$ we have

$$1'_{(s-l)} D_j P^{a_j} D'_j l_{(s-l)} = \begin{cases} s, \text{ if } l = 0 \\ s - l \quad \text{if } a_j = 0 \text{ and } l = 1, 2, ..., s-1 \\ s - 2l, \text{ if } l = 1, 2, ..., a_j \text{ and } a_j \ne s/2 \\ s - l - a_j, \text{ if } l = a_j + 1, a_j + 2, ..., s - a_j - 1 \\ 0, \text{ if } l = s - a_j, s - a_j + 1, ... \text{and } a_j \ne 0 \text{ or if } l = a_j = s/2 \end{cases}$$

and for $a_j > s/2$ we have

$$l'_{(s-l)} D_j P^{a_j} D'_j l_{(s-l)} = \begin{cases} s, \text{ if } l = 0 \\ s - 2l \quad \text{if } l = 1, 2, ..., s - a_j \\ a_j - l, \text{ if } l = s - a_j + 1, ..., a_j - 1 \\ 0, \text{ if } l = a_j, a_j + 1, ..., s - 1 \end{cases}$$

Next let c_j be the contrast vector represented by one of the columns of the matrix $sI_{(s)} - J_{(s)}$ which span the space of s dimensional contrast vectors, where $I_{(s)}$ is an s dimensional identity matrix and $J_{(s)} = 1_{(s)} 1'_{(s)}$, that is an $s \times s$ matrix of $1's$. Also, let the rows and the columns of the matrix $sl_{(s)} - J_{(s)}$ be numbered as $0, 1, 2, ..., s-1$. Note if l levels were deleted from factor F_j in the preliminary factorial design, then the contrast vector c_j is one of the columns of the matrix ${}^{(s-l)}I_{(s-1)} - J_{(s-l)}$ where $l = 1, 2, ..., s-1$.

Lemma 2:

$$c_j' D_j P^{a_j} D_j' c_j = \begin{cases} s(s-1), \text{ if } a_j = 0 \\ -s, \text{ if } l = 0 \text{ and } a_j \neq 0 \\ (s-l)(s-l-1) \text{ if } a_j = 0 \text{ and } l = 1,2,...,s-1 \end{cases}$$

Next for $a_j \neq 0$ and $l = 1,2,...,s-1$ we have the following results.

Lemma 3: If $a_j \leq s/2$ and $l = 1,2,...,a_j$ (provided $a_j \neq s/2$ when $l = s/2$) or if $a_j > s/2$ and $l = 1,2,...,s-a_j$ then

$$c_j' D_j P^{a_j} D_j' C_j = \begin{cases} -l, \quad \text{for } c_j \in \Omega_1 \text{ or } c_j \in \Omega_2 \text{ and } c_j \notin \Omega_1 \cap \Omega_2 \\ s-2l, \quad \text{for } c_j \in \Omega_1 \cap \Omega_2 \\ -s, \quad \text{otherwise} \end{cases}$$

where $\Omega_1 =$ {the columns $a_j - i = 1,2,..,l$ of the matrix $(s-l)I_{(s-l)} - J_{(s-l)}$ } $\Omega_2 =$ {the columns $s-l-a_j+i$, i =0, 1, 2, ..., $l-1$, $i = 0,1,2,...,l-1$ of the matrix $(s-l)I_{(s-l)} - J_{(s-l)}$ }.

Lemma 4: If $a_j \leq s/2$ and $l = a_j+1, a_j+2,...,s-a_j-1$ then

$$c_j' D_j P^{a_j} D_j' C_j = \begin{cases} -a_j, \quad \text{for } c_j \in \Omega_1 \text{ or } c_j \in \Omega_2 \text{ and } c_j \notin \Omega_1 \cap \Omega_2 \\ s-l-a_j, \quad \text{for } c_j \in \Omega_1 \cap \Omega_2 \\ l-s-a_j, \quad \text{otherwise} \end{cases}$$

where $\Omega_1 =$ {the columns $0,1,2,...,a_j$ of the matrix $(s-l)I_{(s-l)} - J_{(s-l)}$ }, $\Omega_2 =$ {the columns s-l-a_j $+i, i = 0,1,2,..,a_j-1$ of the matrix $(s-l)I_{(s-l)} - J_{(s-l)}$ }.

Lemma 5: If $a_j < s/2$ and $l = s-a_j, s-a_j+1,...,s-1$ or if $a_j = s/2$ and $l = s/2$ then $c_j' D_j P^{a_j} D_j' c_j = 0.$

Lemma 6: If $a_j > s/2$ and $l = s-a_j+1, s-a_j+2,...,a_j-1$ then

$$c_j' D_j P^{a_j} D_j' C_j = \begin{cases} a_j - s, \quad \text{for } c_j \in \Omega_1 \text{ or } c_j \in \Omega_2 \text{ and } c_j \notin \Omega_1 \cap \Omega_2 \\ a_j - l, \quad \text{for } c_j \in \Omega_1 \cap \Omega_2 \\ a_j - l - 2s, \quad \text{otherwise} \end{cases}$$

where $\Omega_1 =$ {the columns $a_j - i, i = 1,2,...,l$ of the matrix $(s-l)I_{(s-l)} - J_{(s-l)}$ },
$\Omega_2 =$ {the columns s-$a_j - l - i, i = 0,1,2,..,s-a_j-1$ of the matrix $(s-l)I_{(s-l)} - J_{(s-l)}$}.

Lemma 7: If $a_j > s/2$ and $l = a_j, a_j+1,...,s-1$ then $c_j' D_j P^{a_j} D_j' C_j = 0.$

4. Loss of Information on Main Effects

Dean (1978) showed that for a given vector c^x, the loss of information $\varphi_x, 0 \leq \varphi_x \leq 1$, due to confounding in blocks is given by

$$\varphi_x = \frac{c^{x'} N K^{-1} N' c^x}{c^{x'} c^x} \tag{4.1}$$

where $\boldsymbol{N}$ is the incidence matrix and $\boldsymbol{K}$ is a diagonal matrix of block sizes.

We shall use the notation

$$d_{a_j} = 1'_{s-l} D_j P^{a_j} D'_j 1_{(s-l)}$$

where

$d_{a_j} = 1'_{s-l} D_j P^{a_j} D'_j 1_{(s-l)}$ is as given in lemma 1 and $d_{a_1 a_2 \ldots a_k} = d_{a_1} \times d_{a_2} \times \ldots \times d_{a_k}$

We shall also write

$$d^*_{a_j} = c'_j D_j P^{a_j} D'_j c_j \tag{4.2}$$

where $c'_j D_j P^{a_j} D'_j c_j$ is as given in lemmas 2, 3, 4, 5, 6, and 7.

We consider deletion designs of the form $s^{n-m_1}(s-l)^{m_1}$ with $b = \lambda s^{n-m}$ blocks of size $k = (1/\lambda) s^{n-m_1} (s-l)^{m_1}$ derived from an s^n generalized cyclic design d_p with

$$k_p = (1/\lambda) s^m \text{ and } b_p = \lambda s^{n-m}$$

where n is the number of factors, m_1 is the order of the deletion design constructed, m is the number of generators and $\lambda = \prod_{i=1}^{m} 1/b_i$ with $b_i = HCF(s, g_i); i = 1, 2, \ldots, m$, as given by John and Dean (1975). We give here two results on loss of information on main effects.

Theorem 2: Loss of information due to confounding in blocks on any factor F_j $(j = 1, 2, \ldots, n - m_1)$ whose levels were not deleted from d_p to obtain d, is given by

$$\varphi_x = \frac{\lambda \sum_{a_1} \sum_{a_2} \ldots \sum_{a_n} w_{a_1 a_2 \ldots a_{j-1} a_{j+1} \ldots a_n} \, d_{a_1 a_2 \ldots a_{j-1} a_{j+1} \ldots a_n} \, d^*_{a_j}}{(s-1)(s-l)^{2m_1} s^{n+m-2m_1}}$$

where

$$w_{a_1 a_2 \ldots a_n} = \begin{cases} 1, & \text{if } \mathbf{a} = a_1 a_2 \ldots a_n \text{ is in the initial block of } d_p \\ 0, & \text{otherwise} \end{cases}$$

Proof: The contrast vector c^x is as given in (2.2) $c_t^{x_t}$ is an $s \times 1$ unit vector for $t \neq j, t = 1, 2, \ldots, n - m_1$; $c_t^{x_t}$ is an $(s-l) \times 1$ unit vector for $t = n - m_1 + 1, n - m_1 + 2, \ldots, n$ and $c_j^{x_j}$ is any of the columns of the matrix $sI_{(s)} - J_{(s)}$. Without loss of generality, let $c_j^{x_j}$ be the *i-th* column of the matrix $sI_{(s)} - J_{(s)}$. Then we have

$$c^{x'} c^x = \left[\{ i s^{j-1} + (s-1)^2 s^{j-1} + (s-i-1) s^{j-1} \} s^{n-m_1-j} (s-l)^{m_1} \right] = (s-1)(s-l)^{m_1} s^{n-m_1} \tag{4.3}$$

But

$$K^{-1} = \frac{\lambda}{s^{m-m_1}(s-l)^{m_1}} I_{(\lambda s^{n-m})} \tag{4.4}$$

From John and Dean (1975), and Chauhan (1989) we have,

$$\begin{aligned} c^{x'} N N' c^{x} &= \sum_{a_1}.\sum_{a_2}...\sum_{a_n} w_{a_1 a_2 ... a_n} c_1^{x_1'} D_1 P^{a_1} D_1' c_1^{x_1} \otimes ... \otimes c_n^{x_n'} D_n P^{a_n} D_n' c_n^{x_n} \\ &= \sum_{a_1}.\sum_{a_2}...\sum_{a_n} w_{a_1 a_2 ... a_n}\, d_{\, a_1 a_2 ... a_{j-1} a_{j+1} ... a_n}\, d^*_{a_j} \end{aligned} \tag{4.5}$$

Therefore using (4.2), (4.3), (4.4) and (4.5) in (4.1) we obtain Theorem 2.

Theorem 3: Loss of information due to confounding in blocks on any factor F_j $(j = n-m_1+1, n-m_1+2, ..., n)$ whose levels were deleted from d_p to obtain d, is given by

$$\varphi_x = \frac{\lambda \sum_{a_1}\sum_{a_2}...\sum_{a_n} w_{a_1 a_2 ... a_{j-1} a_{j+1} ... a_n} d_{a_1 a_2 ... a_{j-1} a_{j+1} ... a_n} d^*_{a_j}}{(s-l-1)(s-l)^{2m_1} s^{n+m-2m_1}}$$

where

$$w_{a_1 a_2 ... a_n} = \begin{cases} 1, \text{if } a = a_1 a_2 ... a_n \text{ is in the initial block of } d_p \\ 0, \text{otherwise} \end{cases}$$

Proof: The contrast vector c^x is as given in (2.2), $c_t^{x_t}$ is an $s \times 1$ unit vector for $t = 1, 2, ..., n-m_1$; $c_t^{x_t}$ is an $(s-l) \times 1$ unit vector for $t \neq j, t = n-m_1+1, n-m_1+2, ..., n$ and $c_j^{x_j}$ is any of the columns of the matrix $(s-l)\, I_{(s-l)} - J_{(s-l)}$. Without loss of generality, let $c_j^{x_j}$ be the *i-th* column. Then we have

$$\begin{aligned} c^{x'} c^x &= \begin{bmatrix} \{is^{n-m_1}(s-l)^{j-n+m_1-l} + \\ (s-l-1)^2 s^{n-m_1}(s-l)^{j-n+m_1-1} + (s-l-i-1)s^{n-m_1}(s-l)^{j-n+m_1-1}\}(s-l)^{n-j} \end{bmatrix} \\ &= (s-l-1)(s-l)^{m_1} s^{n-m_1} \end{aligned} \tag{4.6}$$

But we know that

$$c^{x'} N N' c^x = \sum_{a_1}.\sum_{a_2}...\sum_{a_n} w_{a_1 a_2 ... a_n} d_{a_1 a_2 ... a_{j-1} a_{j+1} ... a_n} d^*_{a_j}$$

cf.(4.4). Therefore using (4.4), (4.5) and (4.6) in (4.1) Theorem 3 follows.

5. Confounding in Deletion Designs

The following results in confounding in generalized cyclic designs are due to John and Dean (1975). The number of degrees of freedom confounded with blocks for any given interaction, α^x is given by

$$Y^x = 1/k \sum_{a_1}\sum_{a_2}...\sum_{a_n} w_{a_1 a_2 ... a_n} \left(\prod_{j=1}^{n} z_{a_j}^{x_j}\right) \tag{5.1}$$

where

$$z_{a_j}^{x_j} = \begin{cases} s-1, & \text{if } a_j = 0 \text{ and } x_j = 1 \\ -1, & \text{if } a_j \neq 0 \text{ and } x_j = 1 \\ 1, & \text{if } x_j = 0 \end{cases}$$

and

$$w_{a_1a_2...a_n} = \begin{cases} 1, \ \textit{if } a = a_1a_2...a_n \text{ is in the initial block of } d_p \\ 0, \ \text{otherwise} \end{cases}$$

If the number of degrees of freedom in (5.1) is zero the interaction is unconfounded with blocks and if it is $\prod_{j=1}^{n}(s-1)^{x_j}$ the interaction is totally confounded with blocks. Consider any interactions between the *n* factors, say the interactions of the factors $F_{i_1}, F_{i_2}, ..., F_{i_r}$. Let

$$A = ((a_{ij}))' \tag{5.2}$$

where a_{ij} is from the *i-th* generator $g_i = a_{i1}a_{i2}...a_{in}, i = 1,2,...,m$ and $j = i_1, i_2, ..., i_r$.

Consider all the $f \times f$ sub matrices contained in the j_1-*th*, j_2-*th*, ... j_f-*th* rows of *A* and let $h_{j_1j_2\cdots j_f}$ be the absolute values of their determinants $(f \le r, f \le m)$. Define H_f as follows

$$H_f = \begin{cases} 1, \ \text{if } f=0 \\ HCF(h_{j_1j_2...j_f} \setminus \{j_1, j_2, ..., j_f\} \subset \{i_1, i_2, ..., i_r\}) \text{ if } 0<f<m. \\ 0, \ \text{if } f > m \end{cases} \tag{5.3}$$

In our case the treatment combinations in the initial block are of the form

$$u_1g_1 + u_2g_2 + ... + u_mg_m (u_i = 0,1,...,k_i - 1; i = 1,2,...,m)$$

The number of treatments in the initial block with i_1-*th*, i_2-*th*, ..., i_r-*th* factors all zero is given by $s^{m-r} w_{i_1i_2\cdots i_r}$ where

$$w_{i_1i_2..i_r} = \begin{cases} \prod_{f=1}^{r} HCF(s, H_f / H_{f-1}) \text{ if } r \le m \text{ and } H_r \neq 0 \\ s^{r-g} HCF(w_{j_1j_2..j_g} \setminus \{j_1, j_2, ..., j_g\} \subset \{i_1, i_2, ..., i_r\}) \text{ if } g < r \le m. \\ s^{r-g} HCF(w_{j_1j_2..j_m} \setminus \{j_1, j_2, ..., j_m\} \subset \{i_1, i_2, .., i_r\}, \ \text{if } r > m \end{cases} \tag{5.4}$$

where g is such that $H_g \neq 0$ and $H_{g+1} = H_{g+2} = ... = 0.$ and $g = 1$ if $H_1 = 0$. Let Y^x be denoted by $Y_{j_1j_2..j_h}$ where *x* has the j_1-*th*, j_2-*th*, ..., j_h-*th* digits unity and the remainder zero. Then it can be shown that for the interactions of the factors $F_{i_1}, F_{i_2}, ..., F_{i_r}$ the number of degrees of freedom confounded with blocks is given by

$$Y_{i_1i_2...i_r} = w_{i_1...i_r} - \sum_{g=1}^{r-1} (Y_{j_1j_2..j_g} \setminus \{j_1, j_2, .., j_g\} \subset \{i_1, i_2, .., i_r\}) - 1 \tag{5.5}$$

We now give the following results.

Theorem 4: For the generalized cyclic designs of theorem 1, all the main effects are estimable with full efficiency.

Proof: For the main effect of factor $F_j, j = 1, 2, ..., n$ we have, using (5.3), $H_1 = 1$ and $w_j = 1$ by (5.4). Thus, using (5.5), $Y_j = 0$.

Theorem 5: All the *r*-factor interactions among any number of the first $(n - m + 1)$ factors, and all the *r*-factor interactions among at least two of the first $(n - m + 1)$ factors, and any number of the last $(m - 1)$ factors, are partially confounded with blocks provided $r \le m$.

Proof: For the *r*-factor interaction, $r = 2, 3, .., n - m + 1,$ we have from equation (5.4)

$$w_{j_1 j_2 \cdots j_r} = s^{r\text{-}1}$$

and thus equation (5.5) yields

$$0 \ne Y_{j_1 j_2 \cdots j_r} < (s - r)^r$$

Thus all the interactions between at least two of the first $(n - m + 1)$ factors are partially confounded with blocks.

For $r = 3, 4, ..., n,$ the $r \times r$ submatrices of the matrix A corresponding to the *r-th* factor interactions between at least two of the first $(n - m + 1)$ factors and any number of the last $(m - 1)$ factors are singular. Therefore using equation (5.4) we obtain

$$w_{j_1 j_2 \cdots j_r} = s^{r-2}$$

and using equation (5.5) we get

$$0 \ne Y_{j_1 j_2 \cdots j_r} < (s - r)^r$$

We can therefore conclude that all these r factor interactions are partially confounded with blocks. Hence the theorem.

Confounding in deletion designs has been studied by Chauhan (1989). Theorem 6 below is due to her. Let α^x be a given interaction. Then the factors $F_1, F_2, ..., F_n$ or simply $\{1, 2, ..., n\}$ can be partitioned into three mutually exclusive and exshaustive sets Ω_1, Ω_2 and Ω_3 as follows: Ω_1 contains the factors whose levels were not deleted from d_p to obtain d, that is $\{1, 2, ..., n - m_1\}$; Ω_2 contains the factors whose levels were deleted from d_p to obtain d and these factors are not in the factorial space V_x, that is the factors $\{n - m_1 + 1, n - m_1 + 2, ..., n - m_1 + a\}$; Ω_3 contains the factors whose levels were deleted from d_p to obtain d and these factors are in the factorial space V_x, that is factors $\{n - m_1 + a + 1, n - m_1 + a + 2, ..., n\}; a = 0, 1, 2, ..., m_1$. We shall write the factorial space V_x as $V(j_1, j_2, ..., j_r)$ if $x_{j_1} = x_{j_2} = ... = x_{j_r} = 1$ and all other x_j 's are zero, where $\{j_1, j_2, ... j_r\} \subseteq \{1, 2, ..., n\}$. Let $\{i_1, i_2, ..., i_r\} \subset \{1, 2, ..., n - m_1\}$, then we have the following theorem.

Theorem 6: **(**Chauhan (1989)). Let the contrast vector $c^x \in V(i_1, i_2, ..., i_r, n - m_1 + a + 1,\ n - m_1 + a + 2, ..., n)$ and let

$$c_p^x = D'c^x \tag{5.6}$$

then $c_p^x \in \oplus \; V_p(i_1,i_2,...,i_r,g,n-m_1+a+1,n-m_1+a+2,...,n)$ where $g \in \mathrm{P}(\Omega_2)$; that is g belongs to the power set of Ω_2 and $\oplus$ denotes the direct sum, where $D = D_1 \otimes D_2 \otimes ... \otimes D_n$ again where D_j is as defined in lemma 1.

We now state the following results:

Theorem 7: For the deletion designs of order m derived from generalized cyclic designs of theorem 1, all the main effects of the first $(n-m)$ factors are partially confounded with blocks, while all the main effects of the last m factors are fully estimable.

This makes it possible to proof the following theorem.

Theorem 8: If $m=1$, then for the deletion designs of the form $s^{n-1}(s-l)$ the main effects of the first $(n-1)$ factors are partially confounded with blocks and the loss of information on factor F_j $(j=1,2,...,n-1)$ is given by

$$\varphi_x = \frac{l}{(s-1)(s-l)}$$

provided $l < s/2$.

Proof: From theorem 7 we know that the main effects of the first $(n-1)$ factors are partially confounded with blocks.

$$\varphi_x = \frac{\sum_{a_1}\sum_{a_2}...\sum_{a_n} w_{a_1a_2...a_n} d_{a_1a_2...a_{j-1}a_{j+1}...a_n} d^*_{a_j}}{(s-1)(s-l)^2 s^{n+m-2}} \tag{5.7}$$

using theorem 2. But by lemma 1, if $0 \neq l/2$ then

$$d_{a_n} = \begin{cases} s-l, & if \;\; a_n = 0 \\ s-2l, & if \;\; l \le a_n \;\; or \;\; a_n + l \le s \\ s-l-a_n, & if \;\; a_n < l \;\;\; or \;\;\; if \;\; a_n + l > s \end{cases} \tag{5.8}$$

Therefore due to the nature of the treatment combinations in the initial block, lemma 2 and equations (5.7), (5.8) yield:

$$\varphi_x = \frac{s^{n-2}s(s-1)(s-l) - s^{n-2}s2\sum_{a_n=1}^{l-l}(s-l-a_n) - s^{n-2}s(s-2(l-1)-1)}{(s-1)(s-l)^2 s^{n-1}} \quad = \frac{l}{s-l}$$

as required.

6. Concluding Remarks

These designs give us a simpler method of constructing asymmetrical factorial designs in incomplete blocks. We note that confounding patterns are easily determined from the information gathered from the preliminary factorial designs. Expressions for loss of information in terms of the number of levels, *s*, of the factors in the preliminary design and the number of levels, *l*, deleted from '*j*' factors have been derived.

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