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Characterizations of K- Semimetric Spaces

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تمييز الفضاءات شبه المترية من الصنف K

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خلاصة : نقوم في هذا البحث بكتابة الفضاءات شبه المترية من الصنف K بطرق مختلفة ومتكافئة.

ABSTRACT: In this paper, we prove, for a space X, the following are equivalent:
1. X is a $\omega\Delta_1$ space with a regular-G_δ-diagonal,
2. X is a $\omega\Delta_2$ space with a regular-G_δ-diagonal,
3. X is a semi-developable space with G_δ (3) -diagonal,
4. X is a $\omega\Delta_1$-space with a G_δ(3)-diagonal,
5. X is a $\omega\Delta_2$ -space with a G_δ(3)-diagonal,
6. X is a q, β-space with a G^*_δ (2)-diagonal,
7. X is a semi-developable space with G^*_δ (2)-diagonal,
8. X is a semimetrizable, c-stratifiable space,
9. X is a c-Nagata β-space,
10. X is a K-semimetrizable.

KEYWORDS: $\omega\Delta$ - space,sSemi- developable space, K -semimetrizable space, β -space, G_δ^* (2)-diagonal, G_δ (3)-diagonal, regular-G_δ -diagonal, semi-stratifiable, c -semi-stratifiable.

1. Introduction

A space X is semimetrizable if there exists a real valued function d on $X \times X$ such that

1. $d(x,y) = d(y,x) \geq 0$.
2. $d(x,y) = 0$ if and only if $x = y$..
3. for $M \subset X, x \in \overline{M}$ if and only if $d(x,M) = \inf\{d(x,y) : y \in M\} = 0$. If in addition, d satisfies.
4. $d(H,K) > 0$ whenever H and K are disjoint compact subsets of X, then X is said to be $K-$semimetrizable (Arhangel'skii, 1966) .

Let $\{G_n\}_{n\in\mathrm{N}}$ be a sequence of covers of a space X .

1. Suppose $\{G_n\}_{n\in\mathrm{N}}$ satisfies the following property: if, $x_n \in st(x,G_n)$, then the sequence $\langle x_n \rangle$ has a cluster point.

(a) If, for each $n \in \mathrm{N}$, G_n is an open cover of X , then X is called a $\omega\Delta$-space (Borges, 1968).
(b) If, for each $n \in \mathrm{N}$, $st(x, G_n)$ is an open subset of X , then X is called a $\omega\Delta_1$-space (Gittings, 1975).
(c) If, for each $n \in \mathrm{N}$, $x \in$ Int $st(x, G_n)$, then X is called a $\omega\Delta_2$–space (Gittings, 1975).
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2. If for each $x \in X$, $\{st(x, G_n)\}_{n\in\mathrm{N}}$ is a local base at x , then X is called a semi-developable space. If in addition, for each $n \in \mathrm{N}$, $st(x, G_n)$ is an open subset of X , then X is called a semi-developable space.
3. If, for each $n \in \mathrm{N}$, G_n is an open cover of X and for each $x \in X$,$\bigcap_n st^3(x, G_n) = \{x\}$, then X has a $G_\delta(3)-$diagonal.
4. If, for each $n \in \mathrm{N}$, G_n is an open cover of X and for each $x \in X$, $\bigcap_n \overline{st^2(x, G_n)} = \{x\}$, then X has a $G_\delta^*(2)$- diagonal.
5. If, for each $n \in \mathrm{N}$, $st(x, G_n)$ is an open subset of X and for each $x \in X$, $\bigcap_n \overline{st(x, G_n)} = \{x\}$, then X has a S_2-diagonal.
6. If, for each $n \in \mathrm{N}$, $x \in$ Int $st(x, G_n)$ and for each $x \in X$, $\bigcap_n \overline{st(x, G_n)} = \{x\}$, then X has a α_2-diagonal.
7. If, for each $n \in \mathrm{N}$, G_n is an open cover of X and for any pair of distinct points $x, y \in X$, there exist neighborhoods U and V of x and y, respectively, and $n \in \mathrm{N}$, such that $st(U, G_n) \cap V = \phi$, equivalently, $st(V, G_n) \cap U = \phi$, then X has a regular- G_δ-diagonal.

A COC-map (= countable open covering map) for a topological space X is a function from $\mathrm{N} \times X$ into the topology of X such that for every $x \in X$, and $n \in \mathrm{N}$, $x \in g(n, x)$ and $g(n + 1, x) \subseteq g(n,x)$. A space X is called β-space if X has a COC-map g such that if $x \in g(n, x_n)$ for every $n \in \mathrm{N}$, then the sequence $\langle x_n \rangle$ has a cluster point.

A space X is called $q-$space if X has a COC-map g such that if $x_n \in g(n, x)$ for every $n \in \mathrm{N}$, then the sequence $\langle x_n \rangle$ has a cluster point.

A space X is called *c*-semi-stratifiable (Martin, 1973) (*c*-stratifiable) if there is a sequence $\langle g(n, x) \rangle$ of open neighborhoods of x such that for each compact set $K \subset X$, if $g(n, K) = \bigcup\{g(n, x) : x \in K\}$, then $\bigcap\{g(n, K) : n \geq 1\} = K \left(\bigcap \overline{\{g(n, K) : n \geq 1\}} = K\right)$.
The *COC*-map $g : \mathrm{N} \times X \to \tau$ is called a *c*-semi-stratification (*c*-stratification) of X . A space X is called *c*-Nagata if it is first countable, *c*- stratifiable space.

Throughout this paper, all spaces are assumed to be T_2-spaces unless otherwise stated explicitly. The letter N always denotes the set of all positive integers.

2. Main results

Lemma 1 : Every space with a $G_\delta(3)$- diagonal has a $G_\delta^*(2)$- diagonal.

Proof. Let $\{G_n\}_{n\in\mathrm{N}}$ be a $G_\delta(3)$- diagonal sequence for X . We want to prove that $\bigcap_{n\in\mathrm{N}} st^2(x, G_n) = \{x\}$ for every $x \in X$. Suppose we have $q \in \bigcap_n \overline{st^2(x, G_n)}$. For every open set U such that $q \in U$ and for each $n \in \mathrm{N}$

$$st^2(x, G_n) \cap U \neq \phi.$$

In particular, if $G \in G_n$ is such that $q \in G$ then $st^2(x,G_n) \bigcap G \neq \phi$. So, $q \in st^3(x,G_n)$. As this holds for all n, it follows that $x = q$.

Lemma 2: Any space with a G_δ* (2)-diagonal is a c-stratifiable space.

Proof. Let $\langle G_n \rangle$ be a sequence of open covers of a space X such that $\bigcap_{n\in\mathrm{N}} \overline{st^2(x,G_n)} = \{x\}$. Define a *COC*-map g by

$$g(n,x) = st(x,G_n).$$

We must prove that $\bigcap \overline{g(n,K)} = K$ for any compact subset of X.

Let $p \notin K$. Then, for each $k \in K$, there exists an integer $n(k)$ such that $p \notin \overline{st^2(k,G_{n(k)})}$. Therefore there is an open set $U(k)$ containing p such that $U(k) \bigcap st^2(k,G_{n(k)}) = \phi$. Since K is compact, we can find a finite number of points $k_1,k_2,...,k_r$ of K such that $\{st(k_i,G_{n(k)}) : i = 1,2,...,r\}$ covers K. Let $n = \max\{n(k_i) : i = 1,2,...,r\}$, and $U = \bigcap_{i=1} U(k_i)$. Then

$$U \cap st(k,G_n) = \phi.$$

That is, $U \bigcap g(n,K) = \phi$. This implies $p \notin \overline{g(n,K)}$.

Theorem 1: Every $\omega\Delta_1$-space with S_2-diagonal is an o-semidevelopable space.

Proof. Let $\{G_n\}_{n\in\mathrm{N}}$ be a countable family of covers of a space X illustrating that X is a $\omega\Delta_1$-space. Since X has an S_2-diagonal, there exists a sequence $\langle v_n : n \in N \rangle$ of covers of X such that, for each $x \in X$ and $n \in \mathrm{N}$, $st(x,v_n)$ is an open subset of X and $\bigcap_{n\in\mathrm{N}} \overline{st(x,v_n)} = \{x\}$. For each $n \in \mathrm{N}$, let

$$u_n = \left\{U : U = \left(\bigcap_{i=1}^n G_i\right) \cap \left(\bigcap_{i=1}^n V_i\right), G_i \in G_i, V_i \in v_i, i = 1,2,...,n\right\}.$$

It is easy to see that u_{n+1} refines u_n for all $n \in \mathrm{N}$ and that, for each $x \in X$, $\bigcap_{n\in\mathrm{N}} \overline{st(x,u_n)} = \{x\}$. Furthermore, for each $x \in X$ and $n \in \mathrm{N}$

$$st(x,u_n) = \left(\bigcap_{i=1}^n st(x,G_i)\right) \bigcap \left(\bigcap_{i=1}^n st(x,v_i)\right)$$

and thus $st(x,u_n)$ is an open subset of X. Also it is easy to check that $\langle u_n : n \in \mathrm{N} \rangle$ is a $\omega\Delta_1$-sequence for X.

It remains to show that $\langle u_n : n \in \mathrm{N} \rangle$ is a semi-development for X. Suppose instead that $\langle u_n : n \in \mathrm{N} \rangle$ is not a semi-development for X. Then there is a point x, an open neighborhood W of x, and a sequence $\langle x_n \rangle$ such that for all n, $x_n \in st(x,u_n)$ and $x_n \notin W$. Since $\langle u_n : n \in \mathrm{N} \rangle$ is a $\omega\Delta_1$-sequence for X, the sequence $\langle x_n \rangle$ has a cluster point p. Clearly $p \notin W$ so $p \neq x$. By choice of $\langle v_n : n \in \mathrm{N} \rangle$, there are k in N and a neighborhood V of p such that $V \bigcap st(x,v_k) = \phi$. Now for $n \geq k$, $x_n \in st(x,u_n) \subset st(x,u_k) \subset st(x,v_k)$ so $x_n \notin V$. This contradicts the fact that p is a cluster point of $\langle x_n \rangle$. Thus $\langle u_n : n \in \mathrm{N} \rangle$ is a semi-development for X.

Theorem 2: The following are equivalent for a regular $\omega\Delta_2$-space X :

(1) X is semimetrizable;

(2) X is semi-stratifiable;

(3) X is θ-refinable and has a G_δ-diagonal;

(4) X has a G^*_δ-diagonal;

(5) X has α_2-diagonal.
(6) X is semidevelopable.
Proof. The only implications requiring comment are (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (1). To prove (5) $\Rightarrow$ (6), let $\{G_n\}$ be a countable family of covers of X illustrating that X is a $\omega\Delta_2$-space. Let $\langle v_n : n \in \mathrm{N}\rangle$ be an α_2-sequence for X. Let the sequence $\langle u_n : n \in \mathrm{N}\rangle$ be defined as in the proof of Theorem 2.3. Since for each $x \in X$ and $n \in \mathrm{N}$,
$Intst(x, u_n) = \left(\bigcap_{i=1}^{n} Inst(x, G_i)\right) \bigcap \left(\bigcap_{i=1}^{n} Inst(x, v_i)\right)$,
we have $x \in Inst(x, u_n)$. It follows, exactly as before, that $\langle u_n : n \in \mathrm{N}\rangle$ is a semi-development for X. The implication (6) $\Rightarrow$(1) follows from (Alexander, 1971), Theorem 1.3.

Theorem 3:

For a space X, the following are equivalent:
1. X is a $\omega\Delta_1$-space with a regular -G_δ-diagonal,
2. X is a $\omega\Delta_2$-space with a regular-G_δ-diagonal,
3. X is a semi-developable space with G_δ (3)-diagonal,
4. X is a $\omega\Delta_1$-space with a G_δ (3)-diagonal,
5. X is a $\omega\Delta_2$-space with a G_δ (3)-diagonal,
6. X is a q, β-space with a G^*_δ (2)-diagonal,
7. X is a semi-developable space with G^*_δ (2)-diagonal,
8. X is a semimetrizable, c-stratifiable space,
9. X is a c-Nagata β-space,
10. X is a K-semimetrizable.

Proof. It is clear that $1 \Rightarrow 2$, $3 \Rightarrow 4$, $4 \Rightarrow 5$, $8 \Rightarrow 9$.
The implication $5 \Rightarrow 6$ follows by Lemma 2.5 and since every $\omega\Delta_2$-space is a q, β-space. The implication $6 \Rightarrow 7$ follows by facts every β-space with a G^*_δ-diagonal is a semi-stratifiable space, every q-space with a G^*_δ-diagonal is first countable and every first countable, semi-stratifiable space is a semimetrizable.

The implication $7 \Rightarrow 8$ follows by Lemma 2.2 and since every T_0 semi-developable space is a semimetrizable.

The implication $9 \Rightarrow 8$ follows by facts every c-stratifiable, β-space is semi-stratifiable and every first countable, semi-stratifiable space is a semimetrizable.

$1 \Rightarrow 8$ follows by Lemma 2.2, Theorem 2.3.

For $2 \Rightarrow 3$. Suppose that X is a $\omega\Delta_2$-space with a regular-G_δ-diagonal. Every space with a regular-G_δ-diagonal has a G^*_δ-diagonal. By Theorem 2.4, X is a semi-developable space. Let $\{G_n\}$ be a semi-development and regular- G_δ-diagonal-sequence. To see that G_n satisfies the G_δ(3)-diagonal-sequence, let $x \neq y$ points in X, U and V open sets containing x and y respectively, and n_0 an integer such that if $n > n_0$, then no member of G_n meets both U and V. Let n_1 and n_2 be integers such that $st(x, G_{n_1}) \subset U$ and $st(y, G_{n_2}) \subset V$. $N = \max\{n_0, n_1, n_2\}$. Then no member of G_n meets both $st(x, G_n)$ and $st(y, G_n)$. Thus $y \notin st^3(y, G_n)$.

For $10 \Leftrightarrow 3$. Let $G_n = \{1/n \text{ sphere centered at } x\}$. It is clear that $\langle G_n \rangle$ is a sequence of covers of X and $y \in st(x, G_n)$ if and only if $d(x, y) < 1/n$. Therefore $\langle G_n \rangle$ is a semidevelopment for X. Now let $G_n = \{\text{interior of } 1/n \text{ sphere centered at } x\}$. It is clear that $\langle G_n \rangle$ is a sequence of open covers of X and if $y \in st(x, G_n)$ then $d(x, y) < 1/n$. If there exist distinct points x and

y such that $y \in st^3(x, G_n)$ for all $n \in \mathrm{N}$, then there are sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ such that $x_n \in st(x, G_n), y_n \in st(y, G_n)$ and $y_n \in st(x_n, G_n)$. Let $K_1 = \{x\} \bigcup \{x_n : n \in \omega\}$ and $K_2 = \{y\} \bigcup \{y_n : n \in \omega\}$. We may assume $K_1 \bigcap K_2 = \phi$ with both sets compact. But $d(K_1, K_2) = 0$, a contradiction.

Conversely, let G_n be a semi-development and $G_\delta(3)$-diagonal-sequence for X. Define a semimetric d on X by $d(x, y) = 1/\inf \{j \in \mathrm{N} : x \notin st(y, G_i)\}$. From the definition $x \in st(y, G_n)$ if and only if $d(x, y) < 1/n$. Assume there exist disjoint compacta K and H such that $d(K, H) = 0$. We can find two sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ in K and H respectively, such that $d(x_n, y_n) < 1/n$. Note that X is sequential and T_2 so that $\langle x_n \rangle$ and $\langle y_n \rangle$ have convergent subsequences. Let $\langle x_{n_i} \rangle$ and $\langle y_{n_i} \rangle$ be subsequences of $\langle x_n \rangle$ and $\langle y_n \rangle$ converging to x and y, respectively. Without loss of generality, we may assume $d(x, x_{n_i}) < 1/i$ and $d(y, y_{n_i}) < 1/i$ for each $i \in \mathrm{N}$. Since $d(x_{n_i}, y_{n_i}) < 1/i$, it follows that there is no k such that $y \notin st^3(x, G_k)$. This contradiction completes the proof.

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