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Mapping Methods to Solve a Modified Korteweg-de Vries Type Equation

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ABSTRACT: In this paper, we employ mapping methods to construct exact travelling wave solutions for a modified Korteweg-de Vries equation. We have derived periodic wave solutions in terms of Jacobi elliptic functions, kink solutions and singular wave solutions in terms of hyperbolic functions.

Keywords: Jacobi elliptic functions; Korteweg-de Vries equation; Solitary wave solutions; Travelling wave solutions.

طرق رسم الخرائط لحل معادلة كورتيج دو فريس المعدلة

ادمنان كرشنان

ملخص: نوظف في هذه الورقة طرق رسم الخرائط لإنشاء حلول الانتقال التامة لمعادلة كورتيج دو فريس المعدلة. أوجدنا حلول الموجة الدورية بدلالة دوال جاكوبي التناقصية، وحلول كينك وحلول الموجه الشاذة بدلالة الدوال الزائدية.

كلمات مفتاحية: دوال جاكوبي التناقصية، معادلة كورتيج دو فريس، حلول الموجة الشاذة، حلول الموجة الانتقالية.

1. Introduction

Travelling wave solutions (TWSs) of nonlinear evolution equations have been extensively studied due to their significant applications in mathematical theory and other fields in physical sciences. The advantage of using travelling wave solutions is that the governing partial differential equation (PDE) reduces to an ordinary differential equation (ODE) which makes it easier to solve. Several methods using TWSs have been proposed such as the tanh method [1], the exponential function method [2], the Jacobi elliptic function (JEF) method [3], mapping methods [4-7] etc.

Solitary wave solutions (SWSs) of different types of Korteweg-de Vries (KdV) equations have been a field of intense study in many branches of Physics over several decades [8-11]. A variety of KdV equations have been proposed since the discovery of solitons. Because of the rich structures of the solutions, both mathematicians as well as physicists have been attracted to this field of research.

In this paper, we derive periodic wave solutions (PWSs) of a modified KdV equation in terms of JEFs [12] and deduce their infinite period counterparts in terms of hyperbolic functions such as shock wave solutions and singular wave solutions using mapping methods. The mapping methods employed in this paper give a variety of solutions, such as hyperbolic function solutions, different types of JEFs etc., which other methods cannot do.

2. Mathematical Analysis

In this section, we give an analysis of the mapping methods which will be employed in this paper. The analysis given below is, in general, for a system of PDEs [13] but in this paper we have applied it for a single PDE.

Consider a nonlinear coupled PDE with two dependent variables u and v and two independent variables x and t given by

$$F(u,v,u_t,v_t,u_x,v_x,u_{xxx},v_{xxx},...) = 0, \tag{1}$$

where subscripts denote partial derivatives with respect to the corresponding independent variables, and F is a polynomial function of the indicated variables.

Step 1: Assume that (1) has a travelling wave solution in the form

$$u(x,t) = u(\xi) = \sum_{i=0}^{l_1} A_i f^i(\xi), \quad v(x,t) = v(\xi) = \sum_{i=0}^{l_2} B_i f^i(\xi), \tag{2}$$

where $\xi = x - \lambda t$, A_i, B_i and λ are arbitrary constants, l_1 and l_2 are integers and f^i represents integer powers of f. The first derivative of f with respect to ξ denoted by f' can be expressed in powers of f in the form

$$f'^2 = p f^2 + \frac{1}{2} q f^4 + r, \tag{3}$$

where p, q and r are arbitrary constants. The motivation for (3) was that the squares of the first derivatives of JEFs can be expressed in even powers of themselves.

Step 2: Substituting (2) into (1), the PDE reduces to an ODE. Balancing the highest order derivative term and the highest order nonlinear term of the ODE, the values of l_1 and l_2 can be found.

Step 3: Substituting for u and v and using (3), the ODE gives rise to a set of algebraic equations by setting the coefficients of various powers of f to zero.

Step 4: From the values of the parameters A_i, B_i, p, q and r, the solution of (1) can be derived. Thus a mapping relation is established through (2) between the solution to (3) and that of (1).

It is to be noted that if the values of l_1 and l_2 are integers, we can use the method directly to get a variety of solutions in terms of hyperbolic functions or JEFs. If they are non integers, the equation may still have solutions as rational expressions involving hyperbolic functions or JEFs. The JEFs and their main properties have been described in Appendix.

3. A Modified KdV-type Equation

In this paper, we derive a variety of solutions in terms of JEFs and their infinite period counterparts, which are kink solutions, SWSs or singular SWSs for a modified KdV-type equation derived in [14]. In a study by Wazwaz [15], he has derived kink solutions and single and multi SWSs for this equation using Hirota's method. The equation under consideration is

$$u_{xxt} - u_x u_{xt} - 4u^3 u_t + 4uu_{xxx} - 4u_x u_{xx} - 16u^3 u_x = 0. \tag{4}$$

We search for travelling wave solutions of (4) in the form

$$u(x,t) = u(\xi). \tag{5}$$

Substitution of (5) into (4) gives rise to the ODE

$$(4-\lambda)uu_{\xi\xi\xi} - (4-\lambda)u_\xi u_{\xi\xi} + 4(\lambda - 4)u^3 u_\xi = 0. \tag{6}$$

Dividing by $(4-\lambda)u^2$ and integrating with respect to ξ, we arrive at the equation

$$u_{\xi\xi} - Ku - 2u^3 = 0, \tag{7}$$

where K is the integration constant.

3.1 Mapping Method

Applying the mapping method it can be seen that $l_1 = 1$ and so we assume that (7) has the solution of the form

$$u(\xi) = A_0 + A_1 f(\xi), \tag{8}$$

where f satisfies (3) and A_0 and A_1 are constants to be determined. Cases in which $l_1 > 1$ have been considered in [16].

Substituting (8) into (7) and using (3), we arrive at the following algebraic equations:

$$qA_1 - 2A_1^3 = 0, \tag{9}$$

$$-6A_0 A_1^2 = 0, \tag{10}$$

$$(p-K)A_1 - 6A_0^2 A_1 = 0, \tag{11}$$

$$-KA_0 - 2A_0^3 = 0. \tag{12}$$

Solving these equations, we get

$$A_0 = 0, A_1 = \pm\sqrt{\frac{q}{2}}, K = p. \tag{13}$$

For illustration, we consider the following cases.

Case 1. $p = -2, q = 2, r = 1.$

Now, (7) has the solution $f(\xi) = \tanh\xi$ (see [4,5]). Therefore, (4) has the kink solution

$$u(x,t) = \pm\tanh(x-\lambda t). \tag{14}$$

Case 2. $p = -(1+m^2), q = 2m^2, r = 1.$

Now, (7) has the solutions $f(\xi) = \mathrm{sn}\xi$ and $f(\xi) = \mathrm{cd}\xi$ (see[4,5]). So, (4) has the PWSs

$$u(x,t) = \pm m\,\mathrm{sn}(x-\lambda t) \tag{15}$$

and

$$u(x,t) = \pm m\,\mathrm{cd}(x-\lambda t). \tag{16}$$

As $m \to 1$, the kink solution (14) is recovered from (15), and (16) gives a trivial constant solution.

Case 3. $p = 2-m^2, q = 2, r = 1-m^2.$

In this case, (7) has the solution $f(\xi) = \mathrm{cs}\xi.$ Therefore, (4) has the PWS

$$u(x,t) = \pm\mathrm{cs}(x-\lambda t). \tag{17}$$

As $m \to 1$, (17) gives rise to the singular SWS

$$u(x,t) = \pm\mathrm{csch}(x-\lambda t). \tag{18}$$

Case 4. $p = -(1+m^2), q = 2, r = m^2.$

Here, (7) has the solutions $f(\xi) = \mathrm{ns}\xi$ and $f(\xi) = \mathrm{dc}\xi.$ Thus, (4) has the PWSs

$$u(x,t) = \pm\mathrm{ns}(x-\lambda t) \tag{19}$$

and

$$u(x,t) = \pm\mathrm{dc}(x-\lambda t). \tag{20}$$

As $m \to 1$, (19) leads us to another singular SWS

$$u(x,t) = \pm\coth(x-\lambda t). \tag{21}$$

In the infinite period limit, (20) gives a trivial constant solution. It is worth mentioning that for Case 1, $f(\xi) = \coth\xi$ is also a solution which is a special case of Case 4 when $m \to 1$.

3.2 Modified Mapping Method

This method gives a combination of a hyperbolic function or JEF and its reciprocal as solutions. In this case, we consider the solution of (7) in the form

$$u(\xi) = A_0 + A_1 f(\xi) + B_1 f^{-1}(\xi), \tag{22}$$

where f satisfies (3) and A_0, A_1 and B_1 are constants to be determined. Also, f^{-1} is the reciprocal of f. When $B_1 = 0,$ this method reduces to the mapping method.

Substituting (22) into (7) and using (3), we obtain the following algebraic equations:

$$2rB_1 - 2B_1^3 = 0, \; -6B_1^2A_0 = 0, \tag{23}$$

$$pB_1 - KB_1 - 6B_1A_0^2 - 6B_1^2A_1 = 0, \tag{24}$$

$$-KA_0 - 2A_0^3 - 12B_1A_0A_1 = 0, \tag{25}$$

$$pA_1 - KA_1 - 6A_0^2A_1 - 6B_1A_1^2 = 0, \tag{26}$$

$$-6A_0A_1^2 = 0, qA_1 - 2A_1^3 = 0. \tag{27}$$

These equations lead us to

$$A_0 = 0, A_1 = \pm\sqrt{\frac{q}{2}}, B_1 = \pm\sqrt{r}, K = p - 3\sqrt{2qr}. \tag{28}$$

We now consider some choices for the parameters, as the first three cases of Section 3.1, to obtain the following cases.

Case 1. $p=-2, q=2, r=1.$

Here, (7) has the solution $f(\xi)=\tanh\xi$. So, $u(\xi)=\tanh(\xi)+\coth(\xi)$ satisfies (7) with $K=-8$. Therefore, (4) has the solution

$$u(x,t)=\pm\left[\tanh(x-\lambda t)+\coth(x-\lambda t)\right]. \tag{29}$$

Case 2. $p=-(1+m^2), q=2m^2, r=1.$

So, equation (7) has the solutions $f(\xi)=\mathrm{sn}\xi$ and $f(\xi)=\mathrm{cd}\xi$ with $K=-(1+6m+m^2)$ (see [4,5]). Therefore, (4) has the PWSs

$$u(x,t)=\pm m\,\mathrm{sn}(x-\lambda t)\pm \mathrm{ns}(x-\lambda t) \tag{30}$$

and

$$u(x,t)=\pm m\,\mathrm{cd}(x-\lambda t)\pm \mathrm{dc}(x-\lambda t). \tag{31}$$

As $m\to 1$, the solution (29) is recovered from (30), and (31) gives a trivial constant solution.

Case 3. $p=2m^2, q=2, r=1-m^2.$

In this case, (7) has the solution $f(\xi)=\mathrm{cs}\xi$ with $K=2-m^2-6\sqrt{1-m^2}$. Thus, (4) has the PWS

$$u(x,t)=\pm\mathrm{cs}(x-\lambda t)\pm\sqrt{1-m^2}\,\mathrm{sc}(x-\lambda t). \tag{32}$$

As $m\to 1$, (32) gives rise to the singular SWS (18).

3.3 Extended Mapping Method

This method gives a combination of two different JEFs as solutions. Here, we consider the solution of (7) in the form

$$u(\xi)=A_0+A_1 f(\xi)+B_1 g(\xi), \tag{33}$$

where f satisfies (3), A_0, A_1 and B_1 are constants to be determined, and g satisfies

$$g''=g(c_1+c_2 f^2), g^2=c_3+c_4 f^2. \tag{34}$$

When $B_1=0$, this method also reduces to the mapping method.

The motivation for (34) was the relationship of the second derivative of any JEF with itself and another JEF for some constants c_1, c_2, c_3 and c_4. As in (3), prime denotes differentiation with respect to ξ in (34) as well.

Substituting (33) into (7) and using (3) and (34), we obtain the following algebraic equations:

$$qA_1-2A_1^3-6c_4A_1B_1^2=0, \tag{35}$$

$$c_2B_1-6A_1^2B_1-2c_4B_1^3=0, \tag{36}$$

$$-2(3A_0A_1^2+3c_4A_0B_1^2)=0, \tag{37}$$

$$-12A_0A_1B_1=0, \tag{38}$$

$$pA_1-KA_1-2(3A_0^2A_1+3c_3A_1B_1^2)=0, \tag{39}$$

$$c_1B_1-KB_1-2(3A_0^2B_1+c_3B_1^3)=0, \tag{40}$$

$$-KA_0-2(A_0^3+3c_3A_0B_1^2)=0. \tag{41}$$

Solving these equations, we obtain

$$A_0=0, A_1=\pm\frac{1}{4}\sqrt{3c_2-q}, B_1=\pm\sqrt{\frac{p-K}{6c_3}}, K=\frac{3}{2}c_1-\frac{1}{2}p \tag{42}$$

with the constraint relation

$$3c_2c_3-9qc_3+8c_4(p-K)=0. \tag{43}$$

So, we can deduce from (42) that

$$B_1=\pm\frac{1}{2}\sqrt{\frac{p-c_1}{c_3}}. \tag{44}$$

We consider the case $p=-(1+m^2), q=2, r=m^2$.

Case 1. $c_1 = -m^2, c_2 = 2, c_3 = -1, c_4 = 1.$

So, (7) has the solutions $f(\xi) = \text{ns}\xi$ and $g(\xi) = \text{cs}\xi$ with $K = \frac{1}{2} - m^2$. Therefore, (4) has the PWS

$$u(x,t) = \pm\frac{1}{2}\left[\text{ns}(x-\lambda t) + \text{cs}(x-\lambda t)\right]. \tag{45}$$

As $m \to 1$, (45) gives rise to the singular solution

$$u(x,t) = \pm\frac{1}{2}\left[\coth(x-\lambda t) + \text{csch}(x-\lambda t)\right]. \tag{46}$$

Case 2. $c_1 = -1, c_2 = 2, c_3 = -m^2, c_4 = 1.$

So, (7) has the solutions $f(\xi) = \text{ns}\xi$ and $g(\xi) = \text{ds}\xi$ with $K = \frac{1}{2}m^2 - 1$. Therefore, (4) has the PWS

$$u(x,t) = \pm\frac{1}{2}\left[\text{ns}(x-\lambda t) + \text{ds}(x-\lambda t)\right]. \tag{47}$$

As $m \to 1$, (47) gives rise to the same singular solution (46).

Case 3. $c_1 = -1, c_2 = 2, c_3 = -\frac{m^2}{1-m^2}, c_4 = \frac{1}{1-m^2}.$

In this case, (7) has the solutions $f(\xi) = \text{dc}\xi$ and $g(\xi) = \text{nc}\xi$ with $K = \frac{1}{2}m^2 - 1$. Therefore, (4) has the PWS

$$u(x,t) = \pm\frac{1}{2}\left[\text{dc}(x-\lambda t) + \sqrt{1-m^2}\,\text{nc}(x-\lambda t)\right]. \tag{48}$$

In this case, as $m \to 1$, (48) will lead to a trivial constant solution.

Case 4. $c_1 = -m^2, c_2 = 2, c_3 = -\frac{1}{1-m^2}, c_4 = \frac{1}{1-m^2}.$

Here, (7) has the solutions $f(\xi) = \text{dc}\xi$ and $g(\xi) = \text{sc}\xi$ with $K = \frac{1}{2} - m^2$. Thus, (4) has the PWS

$$u(x,t) = \pm\frac{1}{2}\left[\text{dc}(x-\lambda t) + \sqrt{1-m^2}\,\text{sc}(x-\lambda t)\right]. \tag{49}$$

In this case also, as $m \to 1$, (49) will lead to a trivial constant solution as in the previous case.

In order to illustrate the properties of the travelling wave solutions, we draw some plots of the kink wave solution, as in Figure1, and singular wave solution, as in Figure2.

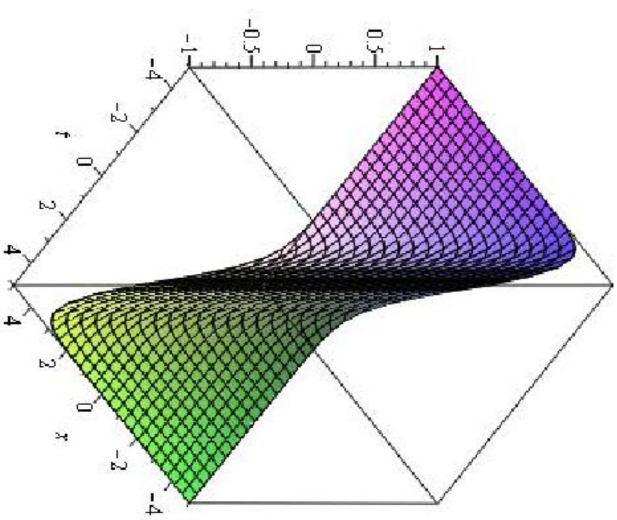

Figure 1. Kink wave solution (14) with $\lambda = 1$.

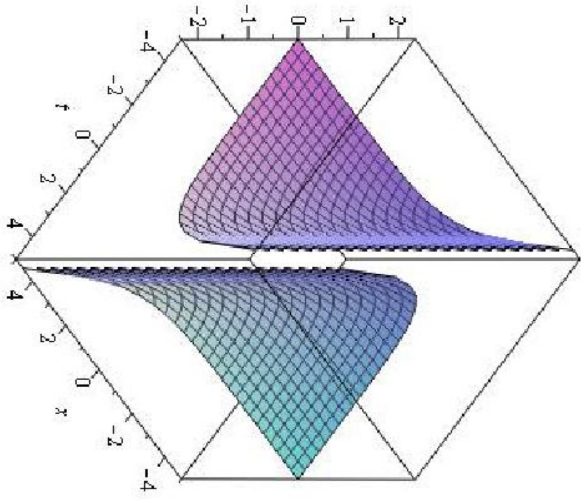

Figure 2. Singular wave solution (18) with $\lambda = 1$.

4. Conclusion

The modified KdV-type equation under consideration was found to have shock wave solutions and singular wave solutions. It was not possible to derive solitary wave solutions in terms of sech functions using the method. It is worth investigating the possibility for such solutions as future research.

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Appendix

Consider the function

$$F(\phi, m) = \eta = \int_0^{\phi} \frac{d\theta}{\sqrt{1-m^2\sin^2\theta}}. \tag{A.1}$$

Letting $t = \sin\theta$, we obtain

$$\eta = \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-m^2t^2)}}. \tag{A.2}$$

This is called Legendre's standard elliptical integral of the first kind.

When $m = 0, \eta = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}x$, where $x = \sin\phi$.

When $m = 1, \eta = \int_0^x \frac{dt}{1-t^2} = \tanh^{-1}x$, where $x = \sin\phi$.

For $0 < m < 1$, we define η as the inverse of a function which is known as Jacobi Sine elliptic function, expressed in the form $\eta = \mathrm{sn}^{-1}x$ so that $x = \mathrm{sn}\eta$ or $\mathrm{sn}\eta = \sin\phi$. Here, m is known as the modulus of the JEFs (see [12]).
We define two other elliptic functions as

$$\mathrm{cn}\eta = \sqrt{1-x^2} = \sqrt{1-\mathrm{sn}^2\eta}, \tag{A.3}$$

and

$$\mathrm{dn}\eta = \sqrt{1-m^2x^2} = \sqrt{1-m^2\mathrm{sn}^2\eta}. \tag{A.4}$$

$\mathrm{cn}\eta$ is known as Jacobi cosine elliptic function and $\mathrm{dn}\eta$ is known as JEF of the third kind.
As $m \to 0$, $\mathrm{sn}\eta \to \sin\eta, \mathrm{cn}\eta \to \cos\eta$ and $\mathrm{dn}\eta \to 1$. As $m \to 1$, $\mathrm{sn}\eta \to \tanh\eta, \mathrm{cn}\eta \to \mathrm{sech}\eta$ and $\mathrm{dn}\eta \to \mathrm{sech}\eta$.
Note that $\mathrm{ns}\eta$, $\mathrm{nc}\eta$ and $\mathrm{nd}\eta$ are the reciprocals of the three JEFs and $\mathrm{sc}\eta$, $\mathrm{cd}\eta$, $\mathrm{ds}\eta$, $\mathrm{cs}\eta$, $\mathrm{dc}\eta$ and $\mathrm{sd}\eta$ are the ratios of the corresponding JEFs.

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