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Boolean Zero Square (BZS) Semigroups

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ABSTRACT: We introduce a new class of semigroups, that we call **BZS** - Boolean Zero Square-semigroups. A semigroup S with a zero element, 0, is said to be a BZS semigroup if, for every $x \in S$, we have $x^2 = x$ or $x^2 = 0$. We obtain some properties that describe the behaviour of the Green's equivalence relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{D}$. Necessary and sufficient conditions for a BZS semigroup to be a band and an inverse semigroup are obtained. A characterisation of a special type of BZS completely 0-simple semigroup is presented.

Keywords: BZS semigroup; Green's relations; Regular; Completely 0-simple; Inverse and band.

أشباه الزمر البولانية ذات المربع الصفري

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الملخص: لقد اوجدنا نوعا جديدا من اشباه الزمر و أطلقنا عليه اسم "اشباه الزمر البولانيه ذات المربع الصفري". شبه الزمرة S الذي يحتوي على صفر 0, يسمى شبه زمرة البولانيه ذو مربع صفري اذا كان كل x في S يحقق اما $x^2 = x$ او $x^2 = 0$. لقد اوجدنا بعض الصفات التي تصف تصرف علاقة جرين التكافؤية $\mathcal{R}, \mathcal{L}, \mathcal{H}$ و$\mathcal{D}$. كما اوجدنا الشروط اللازمة و الشروط الكافية ليكون شبه الزمرة هذا شبه زمره باندية , شبه زمرة معاكسة. كما اوجدنا التوصيف الكامل تنوع خاص من اشباه الزمر البولانيه ذات المربع الصفري وهو شبه الزمرة الصفري البسيط التام.

الكلمات المفتاحية: شبه زمرة البولانيه ذو مربع صفري، علاقة جرين، منتظم، الصفري البسيط التام، معكوس و فرقة.

1. Introduction

We shall use standard semigroup notation, that can be found, for example, in [1]. If S is a semigroup, $E(S)$ represents the set of idempotents of S, and $V(x)$ denotes the set of inverses of an element x in S. We recall that the *natural order* $\leq_n$ on the idempotents of a regular semigroup is defined by

$$e \leq_n f \quad \Leftrightarrow \quad e = ef = fe$$

In [2], Farag and Tucci introduced the notion of a Boolean Zero Square (BZS) ring as an associative ring, not necessarily commutative and not necessarily with identity, such that every non-zero element of R is either idempotent or nilpotent of index 2, that is,

$$(\forall x \in R) \quad x^2 = x \quad \vee \quad x^2 = 0$$

The structure of BZS rings is investigated, in [2] and in [3].
It is possible to find in the literature several papers in Boolean Zero rings, and as a sample we refer to [4] in order to several constructions in this structure.
Here we present a generalisation of these notions to semigroup theory, starting by introducing the following concept, that follows naturally from Ring theory.

Definition. A semigroup S, with element zero 0, is said to be a *BZS semigroup* if, for every element x in S, we have $x^2 = x$ or $x^2 = 0$.

In the ring case, there are more tools available due to the presence of two operations: addition and multiplication. This fact has, as a consequence, that the majority of the results obtained in [2] and [3] cannot be replicated to the semigroup case.

One of the results obtained in [2] is that in a BZS ring, the set of nilpotent elements, is an ideal of the ring. This property does not hold in a general BZS semigroup, as it can easily be seen in Example 2 below. This happens because the result only mentions the multiplicative operation, although its proof uses the additive operation heavily. We obtain in Theorem 8 that in a BZS commutative semigroup the set of its nilpotent elements, is an ideal of the semigroup.

In what follows, unless otherwise stated, S will always denote a BZS semigroup. We denote

$$E = \{x \in S | x^2 = x\} \qquad \text{and} \qquad N = \{x \in S | x^2 = 0\}$$

respectively, the set of *idempotent* elements and the set of *nilpotent* elements of S.

Let us present some basic properties that hold in any such semigroup S.

$$(\forall x \in S) \quad x^3 = x^2 \tag{1}$$

For any $x \in S$, we have two cases to consider:

i) $x^2 = x \quad \Longrightarrow \quad x \cdot x^2 = x \cdot x \quad \Longrightarrow \quad x^3 = x^2$

ii) $x^2 = 0 \quad \Longrightarrow \quad x \cdot x^2 = x \cdot 0 \quad \Longrightarrow \quad x^3 = 0 \quad \Longrightarrow \quad x^3 = x^2$

which proves the result.

$$(\forall x \in S) \quad x^4 = x^2 \quad \text{and} \quad x^2 \in E(S) \tag{2}$$

This follows immediately from (1).

$$E \cap N = \{0\} \tag{3}$$

In fact, if $x \in E \cap N$, we have $x = x^2 = 0$, and the result follows.

$$(\forall x, y \in S) \quad xy \in E\backslash\{0\} \quad \Leftrightarrow \quad yx \in E\backslash\{0\} \tag{4}$$

Let us, assume that $xy \in E\backslash\{0\}$, that is, $(xy)^2 = xy$, with $xy \neq 0$. If $yx = 0$ then,

$$\begin{aligned} yx = 0 \quad &\Longrightarrow \quad x \cdot yx \cdot y = x \cdot 0 \cdot y \\ &\Longrightarrow \quad (xy)^2 = 0 \end{aligned}$$

which is a contradiction, and therefore we can conclude that $yx \neq 0$. Also, if $(yx)^2 = 0$ then, using (1), we have

$$\begin{aligned}(yx)^2 = 0 \quad &\Rightarrow \quad x(yx)^2y = x \cdot 0 \cdot y \\ &\Rightarrow \quad (xy)^3 = 0 \\ &\Rightarrow \quad (xy)^2 = 0 \\ &\Rightarrow \quad xy = 0\end{aligned}$$

which is also a contradiction. So, $(yx)^2 = yx \neq 0$, which means that, $yx \in E\backslash\{0\}$.
Similarly, we prove the reverse implication.

$$(\forall x, y \in S) \quad xy \in N \quad \Leftrightarrow \quad yx \in N \tag{5}$$

If $xy \in N$ then, by (3), $xy \notin E\backslash\{0\}$ and therefore by (4), $yx \notin E\backslash\{0\}$, that is $yx \in N$. The converse implication follows similarly.

Now, let us present some examples to illustrate this concept, which show that they can be found in a wide variety of very well-known classes of semigroups, such as bands, completely 0-simple semigroups and inverse semigroups.
Example 1. Any band, B, with zero is clearly, a BZS semigroup, with $E = B$ and $N = \{0\}$.

Example 2. In a context of ordered semigroup theory, Blyth and McFadden presented in [5] a semigroup which has proved to be very helpful in describing several classes of ordered semigroups. More details of the relevance and properties of this semigroup can also be found in [**6**]. It can be defined by $N_5 = \{u, e, f, a, b\}$ with the following Cayley table:

	u	e	f	a	b
u	u	u	f	f	b
e	e	e	a	a	b
f	u	b	f	b	b
a	e	b	a	b	b
b	b	b	b	b	b

It follows directly from the table that N_5 is a BZS semigroup, b is its zero element, $N = \{a, b\}$ and $E = \{u, e, f, b\}$. This semigroup appears in a different context, as an example of a completely 0-simple semigroup that it is not orthodox. Routine calculations show that it is 0-simple

$$N_5 = N_5 u N_5 = N_5 e N_5 = N_5 f N_5 = N_5 a N_5$$

and, for example, f is a primitive idempotent. Since $ef \neq fe$, we can state that N_5 is not an orthodox semigroup.

Example 3. Consider the completely 0-simple semigroup $S = (I \times G \times \Lambda) \cup \{0\}$ with operation

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if} \quad p_{\lambda j} \neq 0 \\ 0 & \text{if} \quad p_{\lambda j} = 0 \end{cases}$$

$$(i, a, \lambda)0 = 0 = 0(i, a, \lambda) = 00$$

where $G^0 = G \cup \{0\}$ is a zero group, with $G = \langle x \rangle$ an order two cyclic group, I, Λ are non-empty index sets and $P = [p_{\lambda i}]$ is a $\Lambda \times I$ sandwich matrix with entries in G^0, and all the non-zero entries of P are equal to x. Recall that every row and column of P has at least a non-zero entry.

Consider $T = \{(i, x, \lambda) \in S\} \cup \{0\}$ a subset of S, and let $(i, x, \lambda), (j, x, \mu) \in T$. We have the following possibilities for the element $p_{\lambda j}$:

If $p_{\lambda j} \neq 0$, then $(i, x, \lambda)(j, x, \mu) = (i, xp_{\lambda j}x, \mu) = (i, xxx, \mu) = (i, x, \mu) \in T$

If $p_{\lambda j} = 0$, then $(i, x, \lambda)(j, x, \mu) = 0 \in T$

and we can say that T is a semigroup, with the induced semigroup operation. Also, for any $(i,x,\lambda) \in T$, we have that

$$(i,x,\lambda)^2 = \begin{cases} (i,x,\lambda) & \text{if} \quad p_{\lambda i} \neq 0 \\ \\ 0 & \text{if} \quad p_{\lambda i} = 0 \end{cases}$$

Therefore, T is a BZS semigroup.

Example 4. Consider the following set of 2×2 real matrices

$$S = \{I, A, E_{11}, E_{12}, E_{21}, E_{22}, O\} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$$

It is well known that S with the usual matrix multiplication is an inverse semigroup (see, for example [7, Section 7.6, Exercise 1]). S is not a BZS semigroup, since $A^2 = AA = I \neq A, O$.

But if we consider the subset
$$T = \{I, E_{11}, E_{12}, E_{21}, E_{22}, O\},$$

it gives us the following Cayley table

	I	E_{11}	E_{12}	E_{21}	E_{22}	O
I	I	E_{11}	E_{12}	E_{21}	E_{22}	O
E_{11}	E_{11}	E_{11}	E_{12}	O	O	O
E_{12}	E_{12}	O	O	E_{11}	E_{12}	O
E_{21}	E_{21}	E_{21}	E_{22}	O	O	O
E_{22}	E_{22}	O	O	E_{21}	E_{22}	O
O	O	O	O	O	O	O

It follows immediately from the table that T is a subsemigroup of S which is a BZS inverse semigroup, with $E = \{I, E_{11}, E_{22}, O\}$ and $N = \{E_{12}, E_{21}, O\}$.

2. Green's Relations

Let us now obtain some basic properties on the Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{J}$ and $\mathcal{D}$ on a BZS semigroup S.

$$R_0 = L_0 = H_0 = D_0 = J_0 = \{0\} \tag{6}$$

For any $x \in R_0$, we have that $x = x \cdot 1 \in xS^1 = 0S^1 = \{0\}$, which immediately implies that $R_0 = \{0\}$. The other equalities follow similarly.

For $x, y \in E$ or $x, y \in N$ (7)

$$x\mathcal{R}y \quad \Longrightarrow \quad xy\mathcal{R}yx$$

In fact, if $x, y \in E$, then, since $\mathcal{R}$ is a left congruence [1, Proposition 2.1.2],

$$x\mathcal{R}y \quad \Longrightarrow \quad \begin{cases} xx\mathcal{R}xy \\ yx\mathcal{R}yy \end{cases} \quad \Longrightarrow \quad \begin{cases} x\mathcal{R}xy \\ yx\mathcal{R}y \end{cases} \quad \Longrightarrow \quad xy\mathcal{R}yx$$

and, if $x, y \in N$ then

$$x\mathcal{R}y \quad \Longrightarrow \quad \begin{cases} xx\mathcal{R}xy \\ yx\mathcal{R}yy \end{cases} \quad \Longrightarrow \quad \begin{cases} 0\mathcal{R}xy \\ yx\mathcal{R}0 \end{cases} \quad \Longrightarrow_{(6)} \quad xy = 0 = yx \quad \Longrightarrow \quad xy\mathcal{R}yx$$

For $x, y \in E$ or $x, y \in N$ (8)

$$x\mathcal{L}y \quad \Longrightarrow \quad xy\mathcal{L}yx$$

This follows similarly as in (7).

Note that properties (7) and (8) do not hold if one element is in E and the other is in N. To see this, consider N_5 in Example 2, where the $\mathcal{R}$ classes of N_5 are $R_u = \{u, f\}$, $R_e = \{e, a\}$ and $R_b = \{b\}$, while its $\mathcal{L}$ classes are $L_u = \{u, e\}$, $L_f = \{f, a\}$ and $L_b = \{b\}$. From property (7) we have that $e \in E$, $a \in N$, $e\mathcal{R}a$ but $ea = a$ which is not $\mathcal{R}$ related with $ae = b$. Similarly, for property (8).

Theorem 1. *Let S be a BZS semigroup.*
(1) *If $a \in E$, then $R_a \cap E$ is a subsemigroup of S, which is a right zero semigroup. In particular, if $R_a \subseteq E$ then R_a is a right zero semigroup.*
(2) If $a \in N$, then $R_a \cup \{0\}$ *is a subsemigroup of S.*

Proof. (1): It is clear that $R_a \cap E$ is non-empty, since $a \in R_a \cap E$. For any $b, c \in R_a \cap E$, we have,

$$b\mathcal{R}a \quad \text{and} \quad c\mathcal{R}a \quad \Rightarrow \quad b\mathcal{R}c \quad \Rightarrow \quad b = bb\mathcal{R}bc \quad \Rightarrow \quad bc \in R_b = R_a$$

which means that, R_a is a subsemigroup of S. Since b and c are idempotents, we have by [**1**, Proposition 2.3.3], that $bc = c$ and therefore R_a is a right zero semigroup.
(2): It is clear that $R_0 = \{0\}$ is a subsemigroup of S. So, it is enough to consider $a \in N\backslash\{0\}$, that is, $a \neq 0$ and $a^2 = 0$. For $b, c \in R_a$, there exist $x, y, z, w \in S^1$ such that

$$a = bx, \quad b = ay, \quad c = az \quad \text{and} \quad a = cw$$

If, on one hand $b^2 = 0$, then

$$a = bx \quad \Rightarrow \quad ba = b(bx) = b^2x = 0 \cdot x = 0 \quad \Rightarrow \quad ba = 0$$

and

$$bc = b(az) = (ba)z = 0 \cdot z = 0$$

If, on the other hand $b^2 = b$, then by [**1**, Proposition 2.3.3], $bc = c \in R_a$.
Therefore, $R_a \cup \{0\}$ is a subsemigroup of S.
Note that in general, an $\mathcal{R}$ class, R_x, is not a subsemigroup of S. In fact, if we consider the semigroup N_5 of Example 2, and its $\mathcal{R}$ class, $R_e = \{e, a\}$, where $e \in E$ and $a \notin E$, then we have that $ae = b \notin R_e$, which means that, R_e is not a subsemigroup of N_5.

Theorem 2. *Let S be a BZS semigroup.*
(1) *If $a \in E$, then $L_a \cap E$ is a subsemigroup of S, which is a left zero semigroup. In particular, if $L_a \subseteq E$ then, L_a is a right zero semigroup.*
(2) If $a \in N$ then, $L_a \cup \{0\}$ *is a subsemigroup of S.*

Proof. Similar to the proof of Theorem 1.

Like in the note to Theorem 2, we can use Example 2 to illustrate that an $\mathcal{L}$ class of a BZS semigroup is not, in general, a subsemigroup of S.

Theorem 3. *Let S be a BZS semigroup.*
(1) *If $a \in E$, then H_a is a group with only one element.*
(2) *If $a \in S\backslash E$, then $H_a \subseteq S\backslash E$, $(H_a)^2 = \{0\}$ and $H_a \cup \{0\}$ is a subsemigroup of S.*
(3) *If a $\mathcal{D}$ class of S contains an idempotent, all its $\mathcal{H}$ classes are singleton.*

Proof. (1): In fact, by [1, Corollary 2.2.6], H_a is a subgroup of S. We need to prove that H_a has a unique element. For $a = 0$ this is obvious, by (6). Let us now assume that $a \in E\backslash\{0\}$, and consider $b \in H_a$. We have that

$$b\mathcal{H}a \quad \Rightarrow \quad \begin{cases} b\mathcal{R}a \\ b\mathcal{L}a \end{cases} \quad \Rightarrow \quad \begin{cases} bb\mathcal{R}ba \\ ba\mathcal{L}aa \end{cases} \quad \Rightarrow \quad \begin{cases} b^2\mathcal{R}ba \\ ba\mathcal{L}a \end{cases}$$

If $b^2 = 0$ then

$$b\mathcal{H}a \quad \Rightarrow \quad \begin{cases} 0\mathcal{R}ba \\ ba\mathcal{L}a \end{cases} \quad \Rightarrow \quad \begin{cases} ba = 0 \\ a \in L_{ba} \end{cases} \quad \Rightarrow \quad a \in L_{ba} = L_0 = \{0\} \quad \Rightarrow \quad a = 0$$

which is a contradiction. Therefore, we can conclude that $b^2 = b$, and b is an idempotent. Using again [**1**, Corollary 2.2.6], we conclude that $b = a$, and H_a is a singleton subgroup of S.
(2): Let $a \in S\backslash E$, and consider $y \in H_a$, which cannot be equal to 0, by (6). Then,

$$y\mathcal{H}a \quad \Rightarrow \quad \begin{cases} y\mathcal{R}a \\ y\mathcal{L}a \end{cases} \quad \Rightarrow \quad \begin{cases} yy\mathcal{R}ya \\ ya\mathcal{L}aa \end{cases} \quad \Rightarrow \quad \begin{cases} y^2\mathcal{R}ya \\ ya\mathcal{L}0 \end{cases} \quad \Rightarrow \quad y^2 \in R_0 = \{0\} \quad \Rightarrow \quad y \in N$$

and therefore $H_a \subseteq S\backslash E$.
Again, with $a \in S\backslash E$ we have that

$$b\mathcal{H}a \quad \Rightarrow \quad \begin{cases} b\mathcal{R}a \\ b\mathcal{L}a \end{cases} \quad \Rightarrow \quad \begin{cases} ab\mathcal{R}aa \\ ba\mathcal{L}aa \end{cases} \quad \Rightarrow \quad \begin{cases} ab\mathcal{R}0 \\ ba\mathcal{L}0 \end{cases} \quad \Rightarrow \quad \begin{cases} ab = 0 \\ ba = 0 \end{cases}$$

Therefore, for $b, c \in H_a$, we can say that

$$ab = 0, \quad ba = 0, \quad ac = 0 \quad \text{and} \quad ca = 0$$

Since $b\mathcal{H}c$, we have that $b\mathcal{R}c$, and

$$b\mathcal{R}c\mathcal{R}a \quad \Rightarrow \quad cb\mathcal{R}cc\mathcal{R}ca = 0 \quad \Rightarrow \quad cb\mathcal{R}0 \quad \Rightarrow \quad cb = 0$$

from which, we conclude that $(H_a)^2 = \{0\}$ and that $H_a \cup \{0\}$ is a subsemigroup of *S*.
(3): This follows by [1, Lemma 2.2.3] and (1).

Note, that in Theorem 3(2) we verified that, for every $a \in N$, the $\mathcal{H}$ class H_a is a subset of *N*. The same property does not hold for the $\mathcal{R}$ and $\mathcal{L}$ classes. In fact, the semigroup N_5 of Example 2 is such that $a \in N$, but $R_a = \{e, a\} \not\subset N$, as well as $L_a = \{f, a\} \not\subset N$.

Theorem 4. *Let S be a BZS semigroup.*
(1) *If $a \in S\backslash\{0\}$ and $D_a \subseteq E$, then D_a is a subsemigroup of S.*
(2) If $a \in S\backslash E$, then $D_a \cup \{0\}$ *is a subsemigroup of S.*

Proof. (1): For any $a \in S\backslash\{0\}$, we have by (6), that $D_a \neq \{0\}$. The fact that $D_a \subseteq E$ therefore implies that $D_a \subseteq E\backslash\{0\}$.
Considering any $b, c \in D_a$, there exist $d \in S$, such that $b\mathcal{L}d\mathcal{R}c$. By [1, Propositions 2.1.2 and 2.3.3], we have that

$$d\mathcal{R}c \quad \Rightarrow \quad bd\mathcal{R}bc \quad \Rightarrow \quad b\mathcal{R}bc \quad \Rightarrow \quad bc \in R_b \subseteq D_a$$

Thus, D_a is a subsemigroup of *S*.
(2) Consider any $a \in S\backslash E$, that is, $a \neq 0$ and $a^2 = 0$. For any $b, c \in D_a$, there exist $d \in S$, such that $b\mathcal{L}d\mathcal{R}c$, which means, in particular, that $b = xd$ and $c = dy$ for some $x, y \in S^1$.
Also, $d\mathcal{R}c$ implies $b = xd\mathcal{R}xc$ and therefore $xc \in R_b$.
Then,

$$bc = (xd)(dy) = x(dd)y$$

If $d^2 = d$, then $bc = xd^2y = xdy = xc \in R_b \subseteq D_a$.
If $d^2 = 0$, then $bc = 0$.
Thus, $D_a \cup \{0\}$ is a subsemigroup of *S*.

3. Special classes of BZS semigroups

We now devote our attention to obtaining necessary and sufficient conditions for a BZS semigroup *S* to be a band or an inverse semigroup. A characterisation of some BZS completely 0-simple is presented. Also, the commutativity property will be approached.

Theorem 5. *Let S be a BZS semigroup. The following statements are equivalent:*
(1) *S is a band;*
(2) $(\forall x \in S)\ x^3 = x$.

Proof. (1) $\Longrightarrow$ (2): The definition of a band tells us that $x^2 = x$ for all $x \in S$. Then,

$$x^3 = x^2 \cdot x = x \cdot x = x$$

and the result follows.
(2) $\Longrightarrow$ (1): Take an element $x \in S$. Since *S* is BZS, $x^2 = x$ or $x^2 = 0$. If $x^2 = x$, there is nothing to prove. If $x^2 = 0$, then $x = x^3 = x^2x = 0 \cdot x = 0$ which, immediately implies that $x^2 = x$, for every element of *S*, that is, *S* is a band.

In the following Theorem and its proof, we use the identification provided from Rees Theorem [1, Theorem 3.2.3], for a completely 0-simple semigroup *S*. Such *S* is isomorphic to

$$(I \times G \times \Lambda) \cup \{0\},$$

where G is a group, I and Λ are non-empty index sets, and $P = [p_{\lambda i}]$ is a $\Lambda \times I$ sandwich matrix with entries in the zero group $G^0 = G \cup \{0\}$. Also, every row and column of P has at least a non-zero entry. The semigroup operation is defined by

$$(i,a,\lambda)(j,b,\mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if} \quad p_{\lambda j} \neq 0 \\ 0 & \text{if} \quad p_{\lambda j} = 0 \end{cases}$$

Theorem 6. *Let S be a BZS semigroup. The following statements are equivalent:*
(1) *S is a completely 0-simple semigroup with no zero entries in the sandwich matrix;*
(2) *S is a rectangular 0-band.*

Proof. $(1) \Longrightarrow (2)$: Let S be a completely 0-simple semigroup. Considering an arbitrary element x in G, for any $\lambda \in \Lambda$ and $i \in I$, we have that $p_{\lambda i} \neq 0$. Then,

$$(i,x,\lambda)^2 = (i,x,\lambda)(i,x,\lambda) = (i, xp_{\lambda i}x, \lambda) \neq 0,$$

which therefore implies, since S is a BZS semigroup, that $(i,x,\lambda)^2 = (i,x,\lambda)$. Thus,

$$\begin{aligned} (i,x,\lambda)(i,x,\lambda) = (i,x,\lambda) \quad &\Leftrightarrow \quad (i,xp_{\lambda i}x,\lambda) = (i,x,\lambda) \\ &\Leftrightarrow \quad xp_{\lambda i}x = x \\ &\Leftrightarrow \quad p_{\lambda i} = x^{-1} \end{aligned}$$

In particular, if we replace x by the identity element of the group 1_G, we obtain $p_{\lambda i} = 1_G$, and therefore $x^{-1} = 1_G$ which, is equivalent to $x = 1_G$. So, G is the trivial group.
Then, S is isomorphic to $\{(i, 1_G, \lambda): i \in I \text{ and } \lambda \in \Lambda\} \cup \{0\}$, whose elements verify

$$(i,1_G,\lambda)(j,1_G,\mu) = (i,1_G,\mu) \qquad \text{and} \qquad (i,1_G,\lambda) \cdot 0 = 0 = 0 \cdot (i,1_G,\lambda)$$

That is, S is a rectangular 0-band.
$(2) \Longrightarrow (1)$: If S is a rectangular 0-band, then

$$(\forall a \in S)(\forall b \in S\backslash\{0\}) \quad a^2 = a \quad \text{and} \quad aba = a \quad .$$

Then, for any $a, b \in S$ and $b \in S\backslash\{0\}$, we have that, $a = aba \in SbS$ which implies that $S \subseteq SbS$.
Since, the reverse inclusion is always true, we can conclude that S is a 0-simple semigroup.

Also, if in $S\backslash\{0\}$, $a \leq_n b$, then $ab = ba = a$. We have that

$$ab = ba \quad \Rightarrow \quad \begin{cases} aba = baa \\ bab = bba \end{cases} \quad \Rightarrow \quad \begin{cases} a = ba \\ b = ba \end{cases} \quad \Rightarrow \quad a = b$$

which, means that all non-zero idempotents are primitive, and therefore S is completely 0-simple.
Also, if $a, b \in S\backslash\{0\}$, then if $ab = 0$ then $aba = 0 \neq a$, which is a contradiction. So, all the entries of the sandwich matrix are not zero.

It follows from the previous Theorem and its proof that for a BZS semigroup to be completely 0-simple where the sandwich matrix has no zero entries, it is necessary to have a singular group in the middle component of the Rees representation.
In fact, we can say that a BZS semigroup is completely 0-simple where the sandwich matrix has no zero entries if, and only if, it is a completely simple semigroup with a zero adjoined.

Theorem 7. *Let S be a BZS semigroup. S is an inverse semigroup if, and only if, the following conditions hold:*
(1) *S is regular;*
(2) $(\forall x \in S)$ $x'x^2x' = x^2$, *for any inverse* x' *of* x.

Proof. Let S be an inverse semigroup. Any element x in S has a unique inverse denoted by x^{-1}. By [1, Theorem 5.1.1], an inverse semigroup is a regular one, where the idempotents commute. So, by (2), x^2 is an idempotent that, therefore, commutes with xx^{-1} and with $x^{-1}x$. Thus,

$$\begin{aligned} x^4 = x^2 \quad &\Rightarrow \quad x^{-1}x^4 = x^{-1}x^2 \quad \Rightarrow \quad x^{-1}x \cdot x^2 \cdot x = x^{-1}x^2 \quad \Rightarrow \quad x^2 \cdot x^{-1}x \cdot x = x^{-1}x^2 \\ &\Rightarrow \quad x(xx^{-1}x)x = x^{-1}x^2 \quad \Rightarrow \quad x^3 = x^{-1}x^2 \quad \Rightarrow \quad x^3x^{-1} = x^{-1}x^2x^{-1} \\ &\Rightarrow \quad x^2 \cdot xx^{-1} = x^{-1}x^2x^{-1} \quad \Rightarrow \quad xx^{-1} \cdot x^2 = x^{-1}x^2x^{-1} \quad \Rightarrow \quad x^2 = x^{-1}x^2x^{-1} \end{aligned}$$

Conversely let us, assume that (1) and (2) hold. Let e be an idempotent of S and e' any inverse of e. By (2), we have that $e'e^2e' = e^2$, that is, $e' = e$. Thus, we can conclude that each idempotent in S has a unique inverse. Now, considering an element x in S and x', x'' inverses of x, we have that xx' and xx'' are idempotents and inverses of each other, as well as $x'x$ and $x''x$. Thus, $xx' = xx''$ and $x'x = x''x$ and we can deduce that

$$x' = x'(xx') = x'(xx'') = (x'x)x'' = (x''x)x'' = x''$$

The result follows, since by [1, Theorem 5.1.1], a regular semigroup where each element has a unique inverse is an inverse semigroup.

Theorem 8. *Let S be a BZS commutative semigroup. Then,*
(1) *E is a subsemigroup of S;*
(2) *N is an ideal of S;*
(3) *If S is inverse then $x^3 = x$, for every $x \in S$.*

Proof. Consider any elements $x, y \in S$.
(1): If, on one hand, both belong to E, we have $x^2 = x$ and $y^2 = y$, and therefore

$$(xy)^2 = (xy)(xy) = x(yx)y = x(xy)y = (xx)(yy) = x^2y^2 = xy$$

which means that $xy \in E$, and therefore E is a subsemigroup of S.
(2): If, on the other hand, for example $x \in N$, we have that

$$(xy)^2 = (xy)(xy) = x(yx)y = x(xy)y = (xx)(yy) = x^2 \cdot y^2 = 0 \cdot y^2 = 0$$

Thus, $xy \in N$ and we can conclude that N is an ideal of S.

(3): If S is an inverse commutative semigroup, any $x \in S$ has a unique inverse, x^{-1}, and we have by Theorem 7 (2), that

$$x = xx^{-1}xx^{-1}x = x(x^{-1}xxx^{-1}) = x \cdot x^2 = x^3$$

We have seen previously that $x^3 = x$ for all $x \in S$ holds in any BZS semigroup that it is also a band, or a commutative inverse semigroup. It also holds for a BZS completely 0-simple semigroup, where the sandwich matrix has no zero entries. However, this property does not hold for all the BZS semigroups. To see this, let us consider

$$T = \{I, E_{11}, E_{12}, E_{21}, E_{22}, O\}$$

of Example 4, which is an inverse BZS semigroup. Note that we have

$$\begin{array}{llll} I^3 = I, & E_{11}^3 = E_{11}, & E_{22}^3 = E_{22}, & O^3 = O, \\ E_{12}^3 = O \neq E_{12}, & E_{21}^3 = O \neq E_{21} & & \end{array}$$

from which, we can deduce that the mentioned property does not hold in all the BZS semigroups.

4. Conclusion

In this paper, we introduce a new class of ordered semigroups: BZS - Boolean Zero Square semigroups. Several basic properties on Green's relations are obtained. Necessary and sufficient conditions for a BZS semigroup to be a band and to be an inverse semigroup are obtained. A characterisation of a special type of BZS completely 0-simple semigroup is presented.

Conflict of interest

The author declares no conflict of interest.

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