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"Farai Sicome Tòe Amaestrato" (You will Perform, as I Taught You): Notes about Medieval Didactics of Algebra

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Abstract:

The paper studies the medieval tradition of the 9th century al-Khwarizmi's handbook on algebra compared with its Latin translation by Gerard of Cremona (made in Spain, around 1170), later translated in Italian vernacular by an anonymous Florentine abacus master, during the 14th century. This long journey along five centuries and three countries deals accurately with the mathematical contents; by means of analysis of explicit and implied elements in the three works, we also focus on the different historical backgrounds, the social condition of the authors, the cultural, mindset-related and religious obstacles they had to take into consideration, while disseminating these calculation techniques, and, finally, their teaching style.

Keywords:

Algebra; Al-Khwarizmi; Translations; Education; Teaching Style

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Introduction

In this paper, we will outline a part of the history of the 9th century al-Khwarizmi's handbook on algebra, a milestone of the discipline. As a first step, we will expose the content of the book focusing both on its innovative content and on the historical context it was composed in. Then we will consider its Latin translation by Gerard of Cremona, which was made in Spain, around 1170, that is in a completely different cultural environment. Finally, we will study a translation of Gerard's version into Italian vernacular, made by an anonymous Florentine abacus master, during the 14th century, again in a dissimilar historical background. The goal of the work is to highlight the influence of the context on their respective teaching styles, given a substantially identical content.

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Al-Khwarizmi's Handbook

At the beginning of the ninth century AD, in Baghdad the Caliph al-Mansur turned an existing library into an academy, called Bayt al-Hikma (House of Wisdom), where renown scholars of many disciplines were granted access to a huge number of scientific works, mainly translated from Greek. They also contributed to the cultural life of the court and of the reign. Among them, the outstanding scientist al-Khwarizmi, a native of Khoresmia, wrote a handbook on algebra[2] (*Kitab al-hisab al-jabr w'al-muqabalah*, namely *Compendious Book on Calculation by Completion and Balancing*), which, for the great ease of application, would be largely successful, both in the Arabic-speaking Middle East and in the Arabic-European Mediterranean area.

In the preface, the author, after a long praise (*basmala*) in exaltation of God and of the Prophet Muhammad, explains that

> the men of culture born in other times and in peoples who exist no more, have constantly dedicated themselves to writing books on various aspects of the science and different branches of knowledge, both thinking of posterity and of a reward appropriate to their ability, and trusting to obtain reward, wealth and fame, and to respect the language of truth in front of which the endured effort disappears, to unveil the secrets of science and what it secretly holds.
> There is a man who came first to discover what had not been discovered before him and he has left it to posterity; a man who has yet explained what his predecessors had left inaccessible, to clarify the method study or pave the way or bring it closer; a man who finally discovered an error in certain books, and then gathered what was missing, criticized by maintaining a good opinion of the author, without arrogance towards him or pride for his work. (Al-Khwarizmi, 2007, 92-94)[3]

Thanks to the al-Mamun patron, passionate of culture, al-Khwarizmi explains that he felt

> encouraged to compose a brief work on the calculation with [the rules of] completion and reduction; has wanted it to contain what is acute in arithmetic and what is most noble, what necessarily to men serves in cases of inheritance, legacies, divisions, processes and trades and in all mutual relations, or in cases involving the land measurement, channel excavation, geometric calculations, and other content relevant to the calculation and its types [...]. (Al-Khwarizmi, 2007, 94)

The essential purpose of al-Khwarizmi is then to draw up a manual useful for solving the problems of everyday life and this is evident also by observing the magnificent speech style used by the author.

This handbook can be considered the foundation of modern algebra (Rashed 2007), as it offers a complete method for solving equations of first and second degree, thanks to the operations of *al-jabr* (moving a term from one to the other side of the equation, by adding a term of opposite sign to both sides) and *al-muqabalah* (algebraic sum of like terms). The method requires, as a first step, that the equation is reduced, by means of the two operations above, to one of the following six cases (let *a*, *b*, *c* be real and positive):

[2] Al-Khwarizmi's works are many more and deal with arithmetic (*Hisab al-hindi*), astronomy (*Zij*, astronomical tables), geography, and history. The first two spread largely in Mediaeval Europe.
[3] Passages here are taken from the Al-Khwarizmi critical edition edited by Roshdie Rashed with the original text written in Arabic as well as the translation into French. In this article, passages are in author's English translation.

1. $ax^2 = bx$
2. $ax^2 = c$
3. $bx = c$
4. $ax^2 + bx = c$
5. $ax^2 + c = bx$
6. $bx + c = ax^2$

for whose solution the algorithms[4] are provided. As we can see, the method is valid in general and allows the resolution of classes of problems, exemplified by similar equations. As for the style, al-Khwarizmi algebra is completely rhetorical: he does not make use of symbols and the explanations of the various steps are long and redundant.

The whole work structure is the following:

1. Basmala
2. Primitive terms
3. The three simple cases (equations with two terms)
4. The three composed cases (equations with three terms)
5. Demonstration of the rules
6. Operations
7. The 6 problems
8. Other 34 problems
9. The rule of three
10. Measures
11. Heritages
12. Legacies

The content is heavily unbalanced on the side of the practical application: the last two chapters hold most of the treatise and include a very detailed casuistry of possible problems of inheritance and their solutions.

Gerard of Cremona's Latin Translation

The transfer of this method to the European mathematical knowledge is proved not only by the three Latin translations made in Spain between the 12th and the 13th century (Hughes 1982, 1986) by Robert of Chester (1145), Gerard of Cremona (round 1170), and Guglielmo de Lunis (1250), but also by the penetration of the method in the everyday teaching in Mediterranean abacus schools. Leonardo Fibonacci from Pisa studied in one of them: in 1202, in his *Liber Abaci*, he presented (though in Latin) the *summa* of Arabic arithmetic-algebraic knowledge to the benefit of European merchants.

Born around 1114 in Cremona, where he made his first studies, Gerard moved around 1140 to Toledo, to learn Arabic and in search of Ptolemy's *Almagest*. His presence in Spain is documented since 1144 and he died there in 1187 (Björnbo 1905). His production is very extensive: between 70 and 80 books, including at least 12 on astronomy, 17 treatises on mathematics and optics, natural philosophy and 24 medical texts. Gerard's fluency in Latin and his competence in the various fields were such that he was able to convey in Latin not

[4] The word *algorithm* derives from al-Khwarizmi's name transliteration into Latin alphabet, possibly contaminated by the radix of "arithmetica".

only the exact meaning but also the nuances of the Arabic text, according to some sources (Negri Lodrini 1994).

The treatise is divided into 8 chapters with an appendix, with the following content:

1. Introduction: presentation of the numbering system in base 10 and of the algebraic terms

2. Six cases of equations with solved examples

   a. Simple equations
      i. $ax^2 = bx$
      ii. $ax^2 = c$
      iii. $bx = c$

   b. Composed equations
      i. $ax^2 + bx = c$
      ii. $ax^2 + c = bx$
      iii. $bx + c = ax^2$

3. Geometric demonstrations for each composed equation
4. Chapter on multiplication
5. Chapter on sum and difference
6. Chapter of problems (six problems solved)
7. Twelve more problems resolved, starting with the six cases of equations
8. "Rule of Three" with three problems solved

Appendix: 21 solved problems of increasing length and difficulty

The long section about practical applications, present in al-Khwarizmi's is here totally missing.

Vernacular Translations

In the following centuries, vernacular languages began to spread and the merchants' training in mathematics brought this discipline very far from the out-of-date Boetian tradition which had dominated the landscape of the exact sciences in the Middle Ages; these two facts determined the need for vernacular handbooks, in order to easily disseminate these techniques, which in the meanwhile were further developed by other Arabic-speaking mathematicians, such as abu Kamil (IX century) and al-Karaji (X century), or by Europeans, like Fibonacci and many others (Katz 2016).

This is the case of the *Tractatus Algorismi* by Jacopo da Firenze, dating 1327, of the *Liber Abaci* by Paolo Gherardi (early fourteenth century) (van Egmond 1978), or of the *Practica di mercatura* by Francesco di Balduccio Pegolotti in the mid-fourteenth century or again the *Trattato d'Algebra* by Benedetto da Firenze, in the second half of the fourteenth century (Ulivi 2002a): all works in which we can clearly see hints of contamination between the abacus literature and the original Arabic texts (Franci and Toti Rigatelli 1985, Ulivi 2002b, Høyrup 2006, Heeffer 2008a, Høyrup 2008).

More than one scholar, during the fourteenth century, tried to go back to the source of this technique, translating from Latin the algebraic handbook by al-Khwarizmi: hence the translations (both in Florentine vernacular) of Guglielmo de Lunis' version (Rome, Vatican Library, ms. Urb Lat. 291), already studied by Franci (Franci 2003) and Hissette (Hissette

2003), and of Gerard of Cremona's (Florence, BNC, Fond. prin. II.III.198); this last one is the subject of the present work.

The Florence Manuscript

History

As shown at f. 165v, in 1595, the manuscript was owned by the Florentine nobleman Pietro Dini, born in Florence in the second half of the sixteenth century. Pietro showed an interest both in sciences (Formighetti 1991) and in humanities, as he belonged to the *Accademia della Crusca* with the pseudonym of *Pasciuto* (well-fed man) and he was consul at the *Accademia Fiorentina*. In his considerable collection of manuscripts, collected over his entire life, he included, in addition to this manuscript, at least ms. Fond. prin. II.IX.114, containing a copy of the *Trattato d'Algebra* by Benedetto da Firenze. During his career as a prelate, begun thanks to the protection of his maternal uncle Cardinal Ottavio Bandini, he had the opportunity to personally meet Galileo in 1611 and in the Quirinal gardens he witnessed some demonstrations about sunspots. As shown by an intense but short correspondence, dating between February and May of the year 1615, his sincere interest in science, which had urged him to become a friend of Galileo, led him to defend the scientist against the Dominicans' accusations (especially Nicola Lorini's ones) and to do his best for his friend, even turning to Cardinal Inquisitor Bellarmino. Pietro Dini, however, always recommended to the Pisan scientist to be extremely cautious in dealing with the clergy, more than ever about subjects like faith and Bible exegesis. At Dini's suggestion to reduce the Copernican theory to mathematical-astronomical hypotheses without any physical validity (the aim was to avoid consequences), Galileo showed himself politely intransigent: this last fact marked the cooling of relations between the two (Abetti 1945, Banfi 1961, Geymonat 1969).

After the death of its first known owner, the manuscript passed to the Dini family, until its acquisition by the Magliabechiana Library in 1819, when the library, which would later become the *National Library of Florence*, was increasing especially with Italian and European works about sciences (Mannelli Goggioli 1996, Pirolo and Truci 1996, Mannelli Goggioli 2000). In particular, the national (or principal) fund was partly constituted and enriched, at the beginning of the 19th century, with manuscripts from the old Magliabechiana section and then, until 1905, with manuscripts from the libraries of suppressed monasteries; some more manuscripts were purchased or donated.

Content

The ms. consists of eight parts (Mazzatinti 1899, van Egmond 1980, Boncompagni 1862-3): (1) ff. 3r-59v: Anonymous, *Libro d'insegnare arismetricha*, 1390, (2) ff. 60r-65r, Anonymous, *Raccolta di problemi d'abaco*, 1425, 65v-85v: vacua, (3) ff. 86r-107v[5]: Anonymous, *Liber de algiebra ealmuchabila*, 1390, ff. 108r-112v: vacua, (4) ff. 113r-118v: Leonardo Pisano, *Practica Geometriae*, 1390, ff. 119r-123v: vacua, (5) ff. 124r-125V: M.P.I. (?), *Ispermenti di geometria*, 1350, ff. 126r-129v: vacua, (6) ff. 130r-135v: M.P.I. (?), *Calendar*, 1390, ff. 136r-146v: vacua, (7) 147r-158v: Anonymous (but Sacrobosco), *Tractato della spera*, 1380, (8) ff. 159r-165r/a: Anonymous, *Chiose sopra la spera predetta*, 1380, ff. 165r/b-165v: Anonymous, *Sydus venit Saturnalia* (poem in 14 tercets with astronomical content).

The contents of the texts are all intrinsically connected with the world of abacus schools because the works of arithmetic, algebra and geometry (all translated into the vernacular) reveal a didactic aim, proved by the presence of exercises and problems. Texts are accompanied by explanatory drawings: in the text about arithmetic, there are

[5] Also ff. 94r-95r are vacua

representations of counting fingers, of a barrel, and other geometric sketches; geometric figures related to the demonstrations of the three composed rules and hints in the margin, in order to indicate the text of treated problems, are present in the part about algebra.

The algebra translation (3), here attributed to a currently anonymous Florentine Master, holds, as a matter of fact, only ff. 86r-105v and is followed (ff. 106r-107v) by a brief appendix in which the rules for resolution set out above are summarized, to form a sort of booklet, for quick references.

Contrastive Analysis of the Three Works

The Content

The text of the Florentine Master's handbook is acephalic, as it is made evident by the presence in the *incipit* of the anaphoric "Di fuori questi tre modi",[6] referring to the first three cases (the so-called simple cases) of the Arabic treatise. Therefore, we cannot make considerations about the presence of the introductory praise to God (in Arabic *basmala*) and its translation into vernacular. The same applies to the part about the three simple cases and about three of the six problems used by al-Khwarizmi and replicated faithfully by Gerard of Cremona, as examples of an application of the canonical rules. A subsequent gap pertains some problems, which differ completely from both the Latin text and, a fortiori, from the Arabic, reflecting the high degree of freedom in the production of examples and exercises in the abacus schools.

Therefore, to establish a direct comparison of the three versions (Arabic, Latin and Florentine), it is useful to summarize the contents in Table 1.[7]

Chapter	Al-Khwarizmi	Gerard of Cremona	Florentine Master
Basmala	✓	✗	✗ blank
Primitive terms	✓	✓	✗ blank
The three simple cases	✓	✓	✗ blank
The three composed cases	✓	✓	✓
Demonstration of the rules	✓	✓	✓
Operations	✓	✓	✓
The 6 problems	✓	✓	✓ partially blank
Other problems	✓ 34	✓ 12	✓ 10
The rule of Three	✓ 3 examples	✓ 3 examples	✗ 11 examples
Appendix	✗	✓ 21 problems	✓ 17 problems
Measures	✓	✗	✗
Heritages	✓	✗	✗
Legacies	✓	✗	✗

Table 1

[6] Besides these three ways.

[7] As we can see, the Appendix is a part of the treaty not present in al-Khwarizmi's work; it was introduced by Gerard, when he found that different copies of the Arabic text contained different sets of exercises.

Despite the presence of gaps, it is clear that the Florentine Master follows *verbatim* the text of the Latin translation, at least up to the problems, where he draws on types and examples of an already formed abacus tradition, mainly thanks to the work of Fibonacci. His adherence to the source text is also confirmed by the language: the syntax and the lexicon are so punctually based on Gerard's version that illegible sentences, in water- or mechanically damaged pages, can be easily reconstructed by comparison with the corresponding Latin passage.

It is extremely remarkable to highlight that in the chapter about the rule of Three,[8] Gerard takes the same numerical data as his source, though with small variations; the Florentine Master, who omits the theoretical part, still uses the same numerical data, but builds and sets the three problems ("sum to be divided among men" and "price variation related to amount variation") in the merchants' world; he also adds 8 more problems that do not appear neither in the original Arabic text nor in the Latin translation.

As mentioned above, the Florentine Master reserves a certain degree of freedom just for the practical part, that is the text of the problems, which sometimes includes tips, comments or explanations, all missing in the Latin model. To compare the set of problems in the three authors, we must remember that both Gerard and, consequently, the Master, on the one hand, appreciably reduce (from 34 to 10-12) the number of problems, but, on the other hand, add a rich appendix, which fills the gap almost completely up. The full list of the three sets of problems, represented by the corresponding equations, is given here in Table 2.

Problem # in the Arabic text	**Solving equation**	**Position in Latin text**	**Position in Florentine text**
1	$(10-x)x=21$	1	1
2	$(10-x)^2-x^2=40$	2	4
3	$(10-x^2)+x^2+(10-x)-x=54$	3	2
4	$\frac{10-x}{x}+\frac{x}{10-x}=2+\frac{1}{6}$	4	3
5	$\frac{5x}{2(10-x)}+5x=50$	5	5
6	$(10-x)^2=81x$	6-A1[9]	6
7	$nx+(10-n)y=\lvert 10-2n\rvert+\lvert x-y\rvert$ $n\in\mathbb{N}$	-	-
8	$\frac{x}{x+2}=\frac{1}{2}$	7	7
9	$10x=(10-x)^2$	A2	A1

[8] Known also as *Golden Rule*, the rule of Three is a method to solve proportions where three terms are known.

[9] Letter A indicates that the problem is in the Appendix.

10	$\frac{x(10-x)}{10-2x} = 5 + \frac{1}{4}$	8	8
11	$\frac{2}{3} \cdot \frac{x^2}{5} = \frac{x}{7}$	A3	A2
12	$\frac{3}{4} \cdot \frac{x^2}{5} = \frac{4}{5}x$	-	-
13	$4x^2 = 20$	A4	A3
14	$\frac{x^2}{3} = 10$	A5	-
15	$4x^2 = \frac{x}{3}$	A6	A4
16	$x^2x = 3x^2$	A7	A5
17	$4x \cdot 3x = x^2 + 44$	A8	A6
18	$4x \cdot 5x = 2x^2 + 36$	9	9
19	$x \cdot 4x = 3x^2 + 50$	A9	A7
20	$x^2 + 20 = 12x$	A10	A8
21	$\left[x - \left(\frac{x}{3} + 3\right)\right]^2 = x$	10	10
22	$\frac{x}{3} \cdot \frac{x}{4} = x$	A11	A9
23	$\left(\frac{x}{3} + 1\right)\left(\frac{x}{4} + 2\right) = x + 13$	A12	A10
24	$\frac{1 + \frac{1}{2}}{1 + x} = 2x$	11	11
25	$\left(x - \frac{1}{3}x - \frac{1}{4}x - 4\right)^2 = x + 12$	A13	A11
26	$x \cdot \frac{2}{3}x = 5$	A14	A17
27	$\frac{x}{x+2} = \frac{1}{2}$	15	7
28	$\frac{1}{x} - \frac{1}{x+1} = \frac{1}{6}$	12	12
29	$\frac{2}{3}x^2 = 5$	A14	A17

30	$x^2 \cdot 3x = 5x^2$	A16	A12
31	$\left(x^2 - \frac{1}{3}x^2\right) 3x = x^2$	A17	A13
32	$\frac{x^2 - 4x}{3} = 4x$	A18	A14
33	$\sqrt{x^2 - x} + x = 2$	A19	A15
34	$(x^2 - 3x)^2 = x^2$	A20	A16

Table 2

The Epistemological Foundations

A comparison among the three works from an epistemological point of view (Ferrari and Speranza 1994) raises some interesting considerations. Al-Khwarizmi's handbook opens with the so-called *basmala*, a sort of preface, which contains, after the praise to God, the dedication to his patron, the caliph, and an indication of the work purpose and of the recipients. By examining the *basmala* and the text, al-Khwarizmi's work seems characterized by the patterns of change of paradigm (Kitcher 1984):

- *Awareness of the method novelty and of the paradigm change*: "There is a man[10] who came first to discover what had not been discovered before him and left it as a legacy to posterity".
- *Question-answering*: he introduces new solutions (algorithms and problem standardization) to respond to questions that are insoluble with already known techniques;
- *Generalization*: he provides answers to classes of problems and not just to single instances, as is evident from the equations classification.
- *Rigorization*: demonstration of individual passages is introduced.
- *Systematization*: introduction and systematic use of a specific technical vocabulary, as presented here in Table 3.

Arabic term	**Mathematical meaning**	**Literal meaning**
mal	Square of the unknown	Money sum
jidr **shay**	Unknown	Root Thing
adad (mufrad) **dirhem**	Known term	Number (simple) Coins
al-jabr	Shift of a term from an equation member to the other	Completion
al-muqabalah	Algebraic sum of terms having the same degree	Opposition

[10] The author refers to himself.

al-hatt	Normalization to 1 of the coefficient of the square of the unknown	Deposit
al-radd	Normalization to 1 of the coefficient of the square of the unknown (if > 1)	Return
al-ikmal	Normalization to 1 of the coefficient of the square of the unknown (if between 0 and 1)	Perfection
Asamm	Irrational number	Deaf

Table 3

Other features however emerge:

- Care in validating the method by means of geometrical demonstrations, to obtain the consent of the other scientists: he defines himself as "A man who has yet explained what his predecessors had left inaccessible to clarify the method of study or to pave the way or bring access; a man (finally) who discovered an error in some books, and then brought together what was missing, criticized by maintaining a good opinion of the author, without arrogance toward him or pride in his own work";

- importance of the applicative aspects, represented by a set of examples of increasing difficulty, according to explicitly mentioned practical purposes: "I wanted [the work] to contain [...] what people need in cases of inheritance, legacies, divisions, processes and trades and in all dealings with each other or in cases involving the measurement of land, the digging of canals, geometrical computation, and other content relevant to the calculation and its various kinds";

- independence from the front face lesson, despite al-Khwarizmi's speech style, obtained with an abundance of examples and explanations redundancy.

The Latin version by Gerard, being a translation and therefore a restatement of the work contents, shows a substantial but not strict fidelity to the model; he preserves:

- awareness of the novelty of the proposed content and its importance to the Western scientific world;

- independence from the front face lesson;

- largely, the technical vocabulary: *al-jabr* e *al-muqabalah* are transliterated; on the contrary, *mal* (*census*), *jidr* (*radix*), *adad mufrad* (*numerus simplex*), *dirhem* (*dragma*), *al-radd* (*reducere*), *al-ikmal* (*reintegrare*), *asamm* (*surdus*) are translated;

- geometric demonstrations for the three composed cases, confirming the ongoing need to persuade readers of the method validity.[11]

[11] Four centuries later, this strict link between the two sciences is still underlined; see Pedro Nuñez, *Libro de Algebra en Arithmetica y Geometria* (1567), fols. 270–270v. "*Vemos algumas vezes, no poder vn gran Mathematico resoluer vna question por medios Geometricos, y resolverla por Algebra, siendo la misma Algebra sacada de la Geometria, que es cosa de admiracion*" (We see that, in some cases, a great mathematician cannot solve a question by means of geometry, but he succeeds by means of algebra, the same algebra being taken from geometry, which is a thing of wonder).

The Latin translation, however, differs under other respects: it shows, as seen above, a set of numerical examples, that are not the same as its source, probably due to interferences in the Arabic text tradition, handed down in several versions (*in alio tamen haec repperi libro interposita suprascriptis*, Gerard writes, while adding the appendix to the main text); it also implements a de-localization of the parts that seem too closely linked to Islamic culture: the most significant example is the nearly complete disappearance of the praise to God, roughly summarized in "*post laudem Dei et ipsius exaltationem*"; finally, it shows a re-localization, in the Spanish merchants' world, of the problems related to the rule of Three (*capitulum convencionum negociatorum*), of which Gerard retains the same numerical data, but changes the setting. Noteworthy is the case of the first problem, where the word *cafficium* (measure of capacity used in Medieval Spain) appears.

The translation into Florentine vernacular also differs radically from Gerard's text (and consequently from the Arabic), for two reasons: the first one is the presence of such a strong pedagogic vocation, that the manuscript could be considered a true handbook used in the classroom by the teacher, probably an abacus master, and the second one is its indisputable contamination with abacus literature, flourished in parallel with the tradition of Gerard's text and closely associated with the *Liber Abaci* by Leonardo Fibonacci (Heeffer 2008b).

Besides a greater emphasis on calculation, the presence on the margins of the text not only of geometric designs, which already characterized Arabic and Latin specimens but also references (a kind of hash) to the rule enunciation from the problem text, is remarkable. Problems are also easy to find in the page, thanks to the presence, always on the fringes, of tables (similar to matrices), which summarize question data.[12]

One remarkable instance of the pedagogic vocation appears just between the text of the ninth problem and its solution: the Master points out that it is preferable for the student to search the value of the square and not of the unknown, when it is necessary to simplify calculations, by avoiding irrational numbers ("E però in questa questione si.tti vuole amonire algiebra e dartj a intendere che taluolta dej ponere lo numero una cosa e taluolta dej ponere uno censo e però, in questa quistione ched è ora, che.ttu ponghi lo tuo numero uno censo, inperò che.ttu non puoj ponere cosa.[13]"). Such an explicit suggestion is not present, either in al-Khwarizmi's or in Gerard's text, even though both of them apply the rule in the same problem.

In its content, the manuscript shows an increasing (from 3 to at least 11) number of problems whose setting is a market: in the first two issues numerical data is the same as Gerard's, who in turn had taken them over from al-Khwarizmi; they, however, belong to the category of problems of the "sum to be divided among men", typical of abacistic tradition; the third issue has a very simple text, containing references to the Florentine metrology (6 *libbre e 4 once vengono 22 soldi e 4; quanto vengono 16 libbre?*[14]); from the fourth issue on, the Florence-centric point of view, of which the text wrote, emerges without hesitation: the currency is almost always the florin (Goldthwaite and Mandich 1994, Travaini 2003), the standard unit of length measure is the Florentine canna.[15]

The latest examples still belong to the merchants' world: two cases of wool and cloth barter (see Chapter VIII of the *Liber Abaci*), complicated by a deferred payment; the teacher, as in *Liber Abaci*, recommended, for professional ethics, to perform calculations scrupulously

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[12] They are also present in *Liber Abaci*'s manuscripts.

[13] "And in this question, algebra advises and suggests you sometimes let the [searched] number equal to the unknown and sometimes to the square; thus, in the present question, let your [searched] number be equal to the square, as you cannot let it be equal to the unknown".

[14] "*6 pounds and 3 onces cost 22 soldi and 4; how much do 16 pounds cost?*"

[15] The Master shows how to convert it into Pisan, Senese, and Perugian canna; in *Liber Abaci*, chap. VIII, Leonardo Fibonacci teaches how to convert Pisan canna into Sicilian, Genoese, Provençal, Syrian, and Constantinopolitan canna, using the palm as the unit of measurement for the comparison.

"*acciò che niuno non sia inghanato*".[16] Finally, two problems on money lending and three about contracts (4 masters who outsource a job, but have different working rhythms, and a contractor who needs a well, but does not have the whole required sum). The solutions are given in a very detailed and redundant way in the 11 commercial problems. In those from the Latin text, the Master is far more moderate of suggestions: for example, concerning problem A11 (corresponding to #25 in *al-jabr* and to A13 in Gerard's translation), the Master avoids solving the problem in detail, rather like his models, and simply says "*e restaura lo più per lo più e.llo meno per lo meno, sì com'io tòe mostrato e troueraj lo numero 24*"[17].

With contemporary notation, the two solutions are given in Figure 1:

	al-Khwarizmi/Gerard	Florentine Master
Reduction in canonical form	$\left(x-\frac{1}{3}x-\frac{1}{4}x-4\right)^2=x+12$	$\left(x-\frac{1}{3}x-\frac{1}{4}x-4\right)^2=x+12$
	$\left(x-\frac{7}{12}x-4\right)^2=x+12$	$\left(x-\frac{7}{12}x-4\right)^2=x+12$
	$\left(\frac{5}{12}x-4\right)^2=x+12$	$\left(\frac{5}{12}x-4\right)^2=x+12$
	$\frac{25}{144}x^2-\left(3+\frac{1}{3}\right)x+16=x+12$	$\frac{25}{144}x^2+16-\left(3+\frac{1}{3}\right)x=x+12$
	$\frac{25}{144}x^2+4=\left(4+\frac{1}{3}\right)x$	↓
	$x^2+4\left(5+\frac{19}{25}\right)=\left(5+\frac{19}{25}\right)\left(4+\frac{1}{3}\right)x$	↓
	$x^2+\left(23+\frac{1}{25}\right)=\left(24+\frac{24}{25}\right)x$	↓
Solution	$\frac{\left(24+\frac{24}{25}\right)}{2}=12+\frac{12}{25}$	↓
	$\left(12+\frac{12}{25}\right)^2=155+\frac{469}{625}$	↓
	$155+\frac{469}{625}-\left(23+\frac{1}{25}\right)=132+\frac{144}{625}$	↓
	$\sqrt{132+\frac{144}{625}}=11+\frac{13}{25}$	↓
	$x=11+\frac{13}{25}+12+\frac{12}{25}=24$	$x=24$

Figure 1

Linguistic and Expressive Codes

Beyond the inevitable differences due to the employment of three different languages, from the lexical point of view, we can, however, remark something noteworthy (Ferrari 2004, Franci 1996b, 2007, North 1995).

[16] "so that none (=neither the seller nor the buyer) is deceived".
[17] "and balance plus with plus and minus with minus, as I showed you, and you'll find the number 24".

The use of natural language is common to all the three handbooks, both for content presentation and as a support language for theoretical considerations, and finally for anaphoric references to already treated rules or problems. The use of both a textual and graphical language, such as geometry, is also a constant: in the Arabic tradition up to Fibonacci's work, certainly known to the author of the Florentine manuscript, especially Euclidean geometry had become functional to the demonstration of algebraic procedures.

For what pertains technical lexicon, al-Khwarizmi, while recovering from Arabic tradition, uses it in a systematic and consistent way, while Gerard adapts it (as seen, partly) to the Latin language and the Master, working in an already established tradition, translates literally (*algebra ealmuchabila, censo, radice/cosa, numero/dramma; restaurare*). The last interesting issue of the Florentine Master's handbook is the presence of a Latin "explicit", as if he wanted to ennoble what we might call a bibliographic quotation.

The Pragmatics of Mathematics Communication

Since the three handbooks have peculiar features, it appears to be interesting to investigate the differences among the pedagogic styles of the teachers (Putnam 1993, Spagnolo 1998, Speranza 1989). Although we have neither the same technical means as nowadays (such as podcast or video) nor written sources related to these authors (i.e. students' testimony), we can still carry out a study, being confident both in the fact that the teaching-learning relationship is a system, whose dynamics are well-known, and also in the fact that the rules and the medieval way of training have been already studied (Franci 1988, 1996a). Therefore, from the historical study of the teaching style of the three authors and from a contrastive analysis of the treaties, we can say that different teaching contexts clearly emerge (Swetz 1995).

In 9th century Baghdad, al-Khwarizmi, obviously faces an epistemological obstacle: as explicitly said in the *basmala*, he works against prior knowledge, he wants to destroy badly made knowledge, he innovates more than before to empower his audience, and for this purpose he uses a strictly didactic style, and a detailed and meticulous exposition of algorithms, even at the cost of being redundant and repetitive. He writes for practical goals, to endow all literates with basic knowledge of the calculation; otherwise, they wouldn't be able to understand his lessons and to make good use of their content.

In Toledo, nearly three centuries later, Gerard faces epigenetic obstacles: he has to prevent not only religious objections (as a matter of fact, he omits any direct reference to Islam) but also what we could call "noise" in the transmission and in the translation of the text; his source has been delivered to him in multiple handwritten copies, which differ a lot from each other;[18] the translation of the text from Arabic into Latin raises then to Gerard the problem of being both faithful to the original and contemporaneously understandable for an audience, who is different from his predecessor's one. Gerard appears primarily like a bridge between the work of Islamic mathematicians and a socially high and cultured audience (they must know Latin), with basic notions of computing with Arabic numerals, curious to increase their knowledge; his teaching style is still very didactic and redundant.

Finally, in 14th century Florence, when the use of Arabic mathematics is an everyday fact, the Master faces educational obstacles; in abacus schools, where algebra was the last topic of study, reserved only to interested and promising students, he introduces techniques for efficient solutions of trade problems. He can freely adapt the source to his needs,

[18] A striking example of this "noise" could be the inconsistent disposition of the questions in the manuscripts consulted by the translator; Gerard solves this problem by adding an appendix.

entering references in-line (#) and in the margin and the abbreviation *m*[19] standing for multiply. As seen in the discussion, the solution of problems is sometimes drastically simplified: the Florentine author did not need the complete algorithm explanation, because he perfectly mastered the field and he expected his students to remember what he had taught in his previous lessons.

Conclusion

Though the three works share the same content, they differ largely for the audience and the context. In 9th century, Al-Khwarizmi needed to spread a new mathematical approach, and innovative computational techniques, among savants (possibly living at the Caliph's court), whose culture and life were strictly imbued with the study of the Quran. He taught cultured adults how to use algebra to solve very complicated problems raised by the application of the Islamic inheritance laws to real estates or to personal properties. His work can be compared with a remote training course, with theoretical lessons integrated by step-by-step tutorials on a large number of specific questions.

In the 12th century, when Gerard decided to introduce in Europe a significant part of the Arabic mathematical knowledge, he had to purge it from any religious content or reference. His ideal audience was made up by European cultured adults who needed algebra to increase their mathematical knowledge. His work can still be compared with a remote training course, with theoretical lessons, integrated by step-by-step tutorials on a far more limited amount of questions, partly taken from a generic merchants' life.

Finally, the Florentine author's work represents an update of al-Khwarizmi's handbook. His audience was composed by young talented students attending in presence the final course at an abacus school in one of the capitals of European trade. So, he had to adapt the content to his classes, in order to foster their skills in commercial mathematics. Therefore, he simplified or omitted passages, and added problems set in the Italian trade context (e.g. equivalences of local linear measures, like the "channa fiorentina" and the "channa senese"; calculation of interest at the end of the year, which was different from town to town).

In this sense, an interesting evolutive trajectory can be highlighted, across these authors' works and across centuries: from a content- and knowledge-focused pedagogical approach to a more skill-driven one.

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[19] Exactly as what happened to *ad* with @, this *m* is surrounded by a curl; usually this abbreviation referred to the word *moneta/monetae.*

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