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Article

Comparison Between Several Ways of Defining Lebesgue's Integral After Lebesgue

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Abstract:

After the publication in 1904 of Lebesgue's *Leçons*, many mathematicians had the idea of obtaining a simpler and more natural theory of integration, sometimes for a technical or philosophical purpose, sometime for a didactical purpose. Among them Borel, Riesz, Young, Hahn, Pierpoint, Radon, Fréchet, Daniell and in Italy Tonelli, Vitali, Caccioppoli dealt with this fundamental tool. In this paper, their theories are exposed and compared.

Keywords:

Lebesgue's integral; Integration without recourse to Lebesgue measure theory: Integral as limit of integrals of elementary functions; Lebesgue' integral as functional prolongation of Cauchy integral; Abstract integration spaces

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Introduction

In a series of papers (Biacino 2018, 2019, 2020) I examined many aspects of the history of the real function theory during the second halph of the nineteenth and the first decades of the twentieth century. In (Biacino 2018) I dealt with the studies about the evolution of the concept of function after the definitions of Dirichlet and Riemann till the classification by Baire, with the discussions between Baire, Borel and Lebesgue at the beginning of the twentieth century about sets and functions, with their purposes and polemics reported often by their own words. All these manifestations were related as the fertile soil on which the mathematics of the twentieth century grew. In the other previous quoted papers, many technical results about the development of the Lebesgue measure and integration theory and many linked properties and studies are discussed in detail and placed in their proper historical context, underlining, in particular, the contributions of the Italian scholars.

This paper can be considered as the natural conclusion of the previous study, even if it was written almost at the beginning to have a complete picture of the situation under investigation. I expose in it a lot of ideas Lebesgue's genial work engendered, comparing the many ways the mathematicians thought to introduce Lebesgue's theory of integration in a simpler and more natural manner, often reducing the recourse to the measure theory as Riesz, or resorting to the Cauchy- Riemann integral as Borel and Hahn, or adopting a

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functional method as Young and Daniell. It is very interesting to remember that some attempts were made almost contemporarily to the born of Lebesgue theory and were only after proved to be equivalent to it, as for example a first proposal by Young.

There were also attempts with a more emphasized didactical and philosophycal interest, as Tonelli's theory of integral, where the functions that are produced by the choice axiom are banished. His method is clearly linked to Borel's constructive set and function theory. The method used by Pierpoint too has a didactical interest and, as Young's one, allows the recourse to infinite partions constituted by measurable sets of the interval where the function is defined. On the other hand this survey is completed by the exposition of the ways to extend integration to a wider framework, as the methods of Radom and Fréchet, that reflect the tendency to the abstraction pervading some parts of the mathematical world and also other cultural circles in the first twenty years of the twentieth century.

Lebesgue's Measure Theory

It is well known that Riemann introduced the definition of integrable function in his *Habilitationsschrift* presented at Göttingen in 1854; it was published posthumous as a memoir and known only in 1867: in it, Riemann proved the famous characterization of the integrable functions and built a bounded function which is integrable on every bounded interval of the real line but is such that its set of discontinuities is dense in every interval.

Riemann's memoir gave rise to the proliferation of researches and studies about the integrable functions there defined, with a special emphasis about the need to characterize the measure of the set of the discontinuities of such functions: and it was a great merit of Lebesgue's measure theory to solve it by the introduction of the notion of the null measure sets. But before Lebesgue the following theories, strictly linked to the previous problem, emerged: Cantor's set theory (1874-1884); Harnack's (1881) and du Bois-Reymond (1882) content theory; Stolz's and Cantor's (1884) content theory; Peano's (1887) and Jordan's (1893) measure theory; Borel's (1898) measure theory. Only in 1904, by the contemporary introduction of the null measure sets by Lebesgue and of the minimal null extension sets by Vitali it was possible to characterize the Riemann's integrable functions.

During this long period, the statement Borel gave of the problem was fundamental since it allowed us to take a step forward with respect to Peano-Jordan's measure theory by the demand for a countably additive in place of finitely additive measure. Henry Lebesgue (1875-1941) claims in his doctorate thesis, where the theory of measure and integration is formulated and the results already present in many notes on the CR of 1899, 1900 and 1901 are combined, that he was deeply influenced by Borel's formulation of the problem.

Lebesgue's theory was highly innovative in the academic world of the first years of twentieth century and this is why at the beginning it met with difficulties: Fubini remembers that when Lebesgue wrote his doctorate thesis, Picard turned towards Dini, who edited the *Annali di Matematica*, and wrote him that one of his students has devoted himself to the foundation of the Calculus, a subject cultivated in Italy (and particularly by Dini) and therefore it would be better to publish his thesis in Italy (Fubini 1907); clearly Picard did not think highly of his student's work and the previous thesis was published on the *Annali* in 1902. But not long after Picard had to change his mind so that in 1917 in *Les sciences mathématiques en France depuis un demi siècle* he wrote: "*Riemann seemed to have investigated as deeply as possible the idea of the indefinite integral. Lebesgue showed that is not true*" (Picard 1917, 21). As it was predictable Lebesgue's thesis met with strong opposition and its theories turned into the object of lively criticism. So, when Lebesgue in 1902 gave a copy of the just written thesis to Jordan, the old professor said him more or less:

> *Persist in scientific research, and you will experience great joys. But it would help if you learned to enjoy them alone. You will be a source of amazement to others. The learned world will not better understand you: mathematicians have a special place in it, and they do not always read each other* (Lebesgue 1923).[2]

The forecast turned to be not much truthful: indeed , even if Lebesgue was not lacking in cultural opposition during his life, he made a good academic career, teaching at the Cours Peccot of the College of France from 1902 to 1905 and then continued his work, crowned by prizes and honours, at Poitiers, at the Sorbonne and in other universities, becoming at last professor at the Academy of France in 1922.

Lebesgue deals with the problem of the measure of the sets in Chap. VII of his *Leçons*, after a long introduction historical in nature about integral calculus. He begins his exposition giving the axioms for the sets of the real line in Borel's style:

1) two equal sets have the same measure;

2) the measure of the union of a finite number of sets, or of countable many sets, pairwise disjoint, is equal to the sum of the measures;

3) the measure of the interval (0,1) is 1.

Axiom 1) signifies that the measure is invariant with respect to translations; Axiom 2 expresses the fundamental property of countable additivity; Axioms 1), 2) and 3) postulate that the defined measure is an extension of the elementary measure to a larger frame.

After the list of the fundamental requisites a measure has to satisfy, Lebesgue passes to its construction: given a bounded set E, he covers it by a finite or countable set of intervals, pairwise disjoint, the elementary measure of such a cover being the finite or infinite sum of the lengths of the disjoint intervals that compose it; Lebesgue calls *external measure* of E the greatest lower bound of the elementary measures of the previous covers, while the covers vary and, given an interval *(a, b)* containing *E*, defines the *internal measure* of *E* as the measure of *(a, b)* minus the external measure of *(a, b)-E*. Then Lebesgue proves that the internal measure is not greater than the external one and that the two just defined measures are situated between the corresponding Jordan measures. A set is called *measurable* if its internal measure coincides with the external measure (Lebesgue 1904a).

As Riesz observed, "*Lebesgue's measure has the merit of being inclusive without being sensitive*" (Riesz 1920, 191), that is it faces up the delicate themes risen after the introduction of Peano-Jordan's measure, solving the open questions and absorbing in the same time the previous theories, but avoiding the inconveniences they presented: so there are Borel measurable sets that are not Peano-Jordan measurable and vice versa, there are Lebesgue measurable sets that are not Peano-Jordan nor Borel measurable, but all Peano-Jordan and Borel measurable sets are Lebesgue measurable. By the way Lebesgue observes that the set obtained by the Borel's procedure,[3] he calls B measurable, are measurable in the new sense, but, while the Lebesgue measurable sets have the power greater than the continuum (as also the Peano-Jordan measurable sets), the B measurable sets have the power of the continuum, and Lebesgue claims: "but this does not mean defining a non-measurable set B

[2] Persévérez dans le recherche scientifique, vous y éprouverez des grandes joies. Mais il vous faudra apprendre à les goùter solitairment. Vous serez pour le vòtres un sujet d'étonnement. Vous ne serez guère mieux compris du monde savant: les mathématiciens y ont une place a part et ils ne se lisent mème pas toujours les uns les autres (Lebesgue 1923).

[3] That is obtained by finite or countable unions and complements from the real intervals.

is possible… We will only ever encounter measurable sets B" (Lebesgue 1904 a, 109, note 1).[4] It is possible to give examples of Lebesgue measurable sets that are not B measurable using the choice axiom. But, as Lusin showed, it is also possible construct sets of this type without the use of the axiom choice. In other words, it is possible to prove that the class of Lebesgue measurable sets is larger than the class of the B measurable sets without the use of the axiom choice.

After Lebesgue passes to the definition of the measurable functions and proves the theorem:

the limit of a convergent sequence of measurable functions is a measurable function.

He claims that he does not know whether it is possible to name a function that is not B measurable and whether there exist functions that are not measurable. This last question would be answered soon by Vitali (Vitali 1905 d), and Lusin, who gave examples of not Lebesgue measurable functions, obtained applying the axiom of choice. In the sequel, Lebesgue did not admit the not constructive methods by which the previous examples had been built. So in a letter of February 16, 1907 addressed to Vitali he wrote:

> *This kind of idealist reasoning does not have great value in my opinion; but we must consider the information it gives* (Vitali 1984, 457-460).[5]

And also:

> *several authors [Vitali …, Lebesgue …, Ed. van Vleck] have indicated some procedures of formation of not measurable sets, but these procedures suppose some operations that is impossible to carry out or to characterize logically* (Lebesgue 1910 note p. 371).[6]

Robert Solovay proved that it is impossible to define not measurable sets without the use of the choice axiom (Solovay 1970). Then Lebesgue's position consisted in considering as subsets of the real line only the Lebesgue measurable sets. So notwithstanding a large group of mathematicians, the same Lebesgue, Baire, Borel, … , and also in Italy Vitali, Beppo Levi, … , wanted to put in order the concepts of space, time, continuity, in such a way to assign, as Buhl said, superior rights to the logic than to the evidence, yet some of them insisted that it should be possible to give constructive procedures for the new defined entities.

From this point of view the position of Lebesgue was very close to the position of the more severe Borel, who wrote in his introduction to a generalization of Riemann's integral, where he did not want to resort to Lebesgue measure:

> *The idea that guided me is the utility to distinguish between the calculations that can be actually performed and those that cannot be. Only the first ones can be effectively used in the applications of Mathematics. I do not want to say that the applications are the ony purpose of Mathematics, this is very far from my thought … . But I only say that there is a*

[4] mais cela ne veut pas dire qu'il soit possible de définir un ensemble non mesurable B … Nous ne rencontrerons jamais que des ensembles mesurable B" (Lebesgue 1904 a, 109, note 1).
[5] Ce mode de raisonnement idéaliste n'a pas, à mes yeux, grande valeur; mais il est nécessaire de tenir compte des indications qu'il donne (Vitali 1984, letter of February 16, 1907).
[6] divers auteurs [Vitali … , Lebesgue … , Ed. van Vleck …] ont indiqué des procédés de formation d'ensembles non mesurables, mais ces procédés supposent qu'on emploie des opérations qu'on ne sait ni effectuer, ni méme caractériser logiquement (Lebesgue 1910 note p. 371).

great theoretic and practical interest in studying separately the calculable numbers and functions (Borel 1912, 161).[7]

The matter roused much controversy, and many mathematicians were involved in it: with regard to this see (Monna 1972) and also (Biacino 2018). In (Gispert 1995) the positions of Lebesgue, Baire, Borel and other mathematicians with respect to set theory are compared. In order to outline a survey of the measure and integral theory in the history of mathematics between the end of the nineteenth century and the beginning of the twentieth century, see (Nalli 1914) and (Hawkins 2002), where a detailed history of the Riemann and Lebesgue integral is exposed. See also the chapter on integration of (Bourbaki 1963) and (Bottazzini 1990, 1994).

The Theory of Integration by Lebesgue

Lebesgue defines his integral and the concept of a summable function already in his doctorate thesis (Lebesgue 1902); here the exposition he adopts in the not technical paper of 1926 *Sur le développement de la notion d'intégrale* (Lebesgue 1927) will be reported. He starts observing that in the definition of Riemann's integral only partitions of the interval (*a*, *b*) constituted by a finite number of intervals whose lengths tend to zero are considered: in such a way, if the function is discontinuous, we are not sure whether the differences of the bounds of the function in such intervals tend to zero too. The construction proceeds in this way: the interval (a, b) is divided by the points $x_o = a < x_1 < x_2 < \dots < x_n = b$ and for every *i*, z_i is a point of the interval (x_i, x_{i+1}). If $f(x)$ is continuous in (a, b) then the Cauchy's integral of *f* is the limit of the sums $\sum f(z_i)(x_{i+1} - x_i)$ when the greatest length of the intervals of the subdivision tends to zero.

Riemann considers bounded functions and after having carried out the subdivision of (a, b) into partial intervals as before, considers the greatest lower bound m_i and the least upper bound M_i of $f(x)$ in the interval (x_i, x_{i+1}); then he puts:

$$s = \sum m_i(x_{i+1} - x_i);\ S = \sum M_i(x_{i+1} - x_i).$$

The function is called *integrable* by Riemann if, as for the Cauchy sums, the difference *S*-*s* tends to zero when the greatest length of the intervals tends to zero. Some year after Darboux will observe that the two limits of s and S exist whatever the function is (Darboux 1875): they will be called the *lower integral* and the *upper integral* by Volterra in 1881 and they coincide if and only if the function is Riemann integrable, in such a case the common value being the *Riemann integral*.

Lebesgue says: *"The previous definitions are very natural, isn't it?"* but he observes also that they are not useful from a practical point of view. Riemann's definition has the fault that it applies only in very rare occasions and almost by chance. On the contrary let us suppose $f(x)$ is a measurable function[8], and instead of the interval (a, b) let us divide the interval (c, d), where *c* is the greatest lower bound and *d* is the least upper bound of $f(x)$ in (a, b), in intervals whose length is less than ε, inserting the points $y_o = c < y_1 < y_2 < \dots < y_n = d$. Let E_i be

[7] L'idée qui m'a guidé est l'utilité qui me paraìt évidente de distinguer entre les calculs qui peuvent étre réellement effectués et ceux qui ne peuvent pas l'étre. Les premiers, seuls, sont actuellement utilisable dans les applications des Mathématiques. Je ne veux pas dire, bien entendu, que les applications soient l'unique but des Mathématiques; rien n'est plus loin de ma pensée; … Ce que je dis simplement, c'est qu'il y a un grand intérét théorique et pratique à étudier à part les nombres et les fonctions calculables … (Borel 1912, 161).

[8] A real function defined in an interval *I* is *measurable* if, for every real number *a*, the set $\{x \epsilon I : f(x) > a\}$ is measurable.

the set $\{x \in (a, b): y_i \le f(x) < y_{i+1}\}$. E_i is measurable since such is the function $f(x)$. Let u_i whatever point between y_i and y_{i+1} and consider the sum $\sum_{i=0}^{i=n-1} u_i m(E_i)$. It is possible to prove that when ε tends to zero such a sum tends to a finite limit for every bounded measurable function: this limit is by definition the *Lebesgue's integral* of $f(x)$.

Lebesgue gives an example to show his point of view. A tradesman can count his money as he gets it, but he can also proceed in a different way, assembling first all the pieces of a given value and then summing. So he sums m_1 pieces of 10 euros, m_2 pieces of 20 euros, m_3 pieces of 50 euros and so on, obtaining the result

m_1 x 10 + m_2 x 20 + m_3 x 50 + …

Lebesgue observes that the two results are obviously equal because, in this case, we deal with finite sums, but he underlines:

> *for us that have to sum infinitely many indivisibles the difference between the two methods is of capital importance* (Lebesgue 1927, 153).

In an analogous way it is possible to apply the previous procedure to measurable but unbounded functions; in this case however the integral sum may be not convergent. If the integral sums tend to a finite limit the function is called *summable*.

The use for unbounded functions of integral sums is rather difficult. It is for this reason that de la Vallée Poussin introduced another definition of integral for measurable unbounded functions basing it on the notion of "truncated" function (de la Vallée Poussin 1915, 1916). The new integral, equivalent to the Lebesgue's one, is defined in the following way: given a nonnegative function f on a bounded measurable set E, let, for every given natural number n, $f_n(x)$ be equal to $f(x)$ if $f(x) \le n$, and to n if $f(x) > n$. The function f_n is bounded and measurable so it is integrable. Consider the sequence of the integrals of f_n: it is not decreasing and therefore it has a finite or infinite limit. In the first case the function is called *summable* and by definition

$$\int_E f(x)dx = \lim_n \int_E f_n(x)dx.$$

If f is of whatever sign it is called *summable* if its absolute value is summable in the previous sense; in such a case, the function is written as the difference between its positive part $f_1(x)=max\{0, f(x)\}$ and its negative part $f_2(x)=max\{0, -f(x)\}$, which are two measurable and nonnegative functions, they are also both summable and the integral is given by definition by:

$$\int_E f(x)dx = \int_E f_1(x)dx - \int_E f_2(x)dx.$$

Observe that yet in 1905 Vitali had given the notion of "truncated" function in the following way: let f be a function and let p and q be two real numbers such that $p<q$. Let $\{f\}_p^q$ be equal to $f(x)$ if $p \le f(x) \le q$, equal to p if $f(x) < p$, equal to q if $f(x) > q$. But this notion was used by Vitali in another frame (Vitali 1905 b).

The Lebesgue integral is a distributive and additive functional: it has the great merit to be a powerful instrument to carry out difficult calculations allowing the passage to the limit under the integral sign in many interesting cases.

Other Ways to Introduce Lebesgue's Measure and Integral: The Riesz Approach

It is evident from the previous exposition that Lebesgue's integral is strictly linked to measure theory: but this theory was considered by many mathematicians too abstract, not much intuitive and not constructive. Consequently, after a first period in which the new and revolutionary approach to integration penetrated the mathematical world and was absorbed, some attempts to generalize Riemann's theory in a different manner from Lebesgue were made.

The Hungarian mathematician Frigyes Riesz (1880-1956) was interested in Lebesgue's integral since he read the memory that Lebesgue had dedicated to the trigonometric series, but as he relates in (Riesz 1920, 1949), he was also interested in an alternative method to Lebesgue's theory that Borel was the first to expose with a note on the C.R. of 1912, followed in the same year by a memory on the *Journal de Mathematiques* (Borel 1912). Riesz took these papers has an opportunity to expose his own ideas on the same question, ideas very different from Borel's ones: Riesz defined the integral by a functional method using only the notion of null measure set and deducing all the measure theory as a chapter on the integration, the measurable sets being those sets whose characteristic functions are summable.

On the other hand Borel started his treatment by means of the measure theory he firstly had introduced since 1898 and tried to extend the integration to discontinuous unbounded functions generalizing Cauchy's method. He intended in this way to give a foundation of measure and integration theory based on a constructive method. Indeed among other things in the memoir of 1912 the definitions of a constructive number and of a constructive function were given for the first time.

Because of these opposite approaches, a controversy followed where Borel and Lebesgue were involved: even if it went on in a calm tone (Borel in the memoir of 1912 counts Lebesgue among his friends), it revealed two very different way to look up mathematics: while Lebesgue had some reservations about not constructive methods, Borel had these reservations in a more accentuated manner. The controversy finished by the memoir of 1918 by Lebesgue, where the author observes that Borel thinks that:

> *The constructive method is more natural, more simple, more direct and more rapid; it is more general but it makes to know more exactly the significance of the results vitiated by a too great (apparent) generality in my works* (Lebesgue 1918, 192).[9]

As a consequence, Lebesgue refutes Borel's arguments and criticism in the following way:

> *I apologize for the length of this work, but, since I could not find, in the Mémoires of M. Borel, the exposition of his reasons and the explanation of what is a constructive definition, either, I had to examine all the hypotheses I thought* (Lebesgue 1918, 192).[10]

The English mathematician Young also dealt with the same question from different perspectives, in 1905 and 1911, but unfortunately his interesting works passed almost

[9] La méthode constructive serait plus naturelle, plus simple, plus directe et plus rapide; serait plus générale et cependant ferait connaître plus exactement la portée des résultats, entachés d'une trop grand généralité (apparente) dans mes travaux (Lebesgue 1918, 192).
[10] Je m'excuse de la longueur de ce travail; mais, comme je n'ai plus pu trouver, dans les Mémoires de M. Borel, l'exposé des raisons qui motivent son opinion, ni même l'explication de ce qu'est une définition constructive, j'ai dû examiner toutes les hypothéses auxquelles j'ai pensé (Lebesgue 1918, 192).

unnoticed in the first moment. Even so, his functional method, as Riesz's one, was the starting point for the more general theory of the integral exposed by Daniell in 1918.

Other attempts to develop an integration theory without referring to Lebesgue's measure on the whole or partially were made also: for example Sierpinski (1882-1969) in 1916, in order to give a simpler proof of Vitali-Lusin theorem, simplifies the definition of a measurable set, basing it on the fundamental notions of set theory. Precisely he calls a subset *M* of the real line *measurable* if it is such that for every $\varepsilon > 0$ there exist two sets *F* and *N* such that $M = F \cup N$, where *F* is a closed set and *N* can be covered by a sequence of intervals such that the sum of their lengths is less than ε. Sierpinski does not say what is the measure of a measurable set. But, given the notion of a measurable set he can deduce consequently a notion of measurability for functions, essentially in the manner of Lebesgue, obtaining equivalent definitions even if not identical to Lebesgue's ones (Sierpinski 1916).

Riesz approach forms part of the same scheme of things, but with a more well - made argument, since this author, as already said, considers the integration problem in its generality, using of Lebesgue measure theory only the sets whose Lebesgue's measure is zero. His exposition has the great merit to make easier the proofs of the fundamental theorems of integration theory.

He starts calling a function *measurable* if it is defined almost everywhere on a closed interval $[a, b]$ and is the limit of an almost everywhere convergent sequence of step functions.

A *step function* in turn is a function which is constant on each open interval of some partition of $[a, b]$ by a finite number of points $a = x_0 < x_1 < \dots < x_n = b$.

Together with the step functions, whose integrals are obviously known in the usual manner, Riesz considers the class of the functions *f* that can be represented as the limits in the sense of convergence almost everywhere of a monotone increasing sequence of step functions, whose integrals are bounded: such a function can be proved to be almost everywhere finite and measurable in the previous sense.

For such a function the *integral is* the limit of the sequence of the integrals of the step functions whose limit is *f* (obviously Riesz proves that, given *f*, all the previous sequences of integrals have the same limit). A function that is the difference of two functions of the previous class is called *summable* and its integral is the difference of the integrals of the two functions which represent it.

It is possible to prove that the class of the Riesz measurable functions coincides with the class of the Lebesgue measurable functions and so for the class of summable functions and for the integrals. This is not surprising since every Lebesgue measurable set can be obtained from a Borel set by adding or subtracting a set of null measure.

Young's First Methods of Integration

William Henry Young (London 1863- Lausanne 1942) proposed an original approach to Lebesgue' integral where the upper and lower semicontinuous functions play a fundamental role. In 1905 Young publishes *On Upper and lower integration*: the paper, arrived at the editorial office in 1904, January, was written when Young was not informed yet of Lebesgue's work, but, as the author claims in the introduction of the paper, once he had known it, he could notice his paper agreed perfectly with it, having moreover the merit to consider functions of more variables.

The author wants to calculate the upper or lower integral, in the sense of (Darboux 1875), of a bounded discontinuous function of one or more variables by reducing it to an ordinary integration: in the scheme of things of Baire this problem is faced essentially for the bounded upper and lower semicontinuous functions, which, introduced by Baire in his thesis

(Baire 1899), are Lebesgue integrable; the analogous problem for whatever discontinuous function is reduced to the previous case. Young reduces the upper integral to an ordinary Riemann's integration in the following way: given a bounded function $f(x)$ defined on a bounded and simply connected region D of the n-dimensional space, the upper integral of $f(x)$ extended to D is given by:

$$K|D|+\int_{K}^{K'} I(k)dk,$$

where $|D|$ is the content (the exterior Jordan's measure) of D, $I(k)$ is the content of the set of the points of D such that $f(x)\geq k$, K and K' are the greatest lower bound and the least upper bound respectively of $f(x)$ in D (Young 1905a, 57). Notice that $I(k)$ is a not increasing function of k and therefore is Riemann integrable. Moreover, if the function $f(x)$ is upper semicontinuous then $I(k)$ is the exterior Jordan's measure of a close set and therefore coincides with Lebesgue's measure; so in this case the upper integral coincides with Lebesgue's integral.

Analogously Young deals with the lower integral of a lower semicontinuous function.

In order to define the integral in general Young uses a theorem given by Baire in the following way. Given a bounded discontinuous function $f(x)$ defined on a region D it is possible to associate $f(x)$ with three other functions:

- the function that associates every point of D with the maximum limit of $f(x)$ in such a point;
- the function that associates every point of D with the minimum limit of $f(x)$ in such a point;
- the oscillation of $f(x)$, that is the function that associates every point of D with the difference of the values of the first and the second function in that point.

The first and the third functions are upper semicontinuous, while the second one is lower semicontinuous.

The result proved by Young is the following:

> *The upper integral of a discontinuous function coincides with the upper integral of its maximum limit function, that is integrable in the previous sense, since this function is upper semicontinuous; analogously the lower integral coincides with the lower integral of its minimum limit function* (Young 1905a, 56).

Again in 1905 another paper on integration by Young appeared on the *Philosophical Transactions* (Young 1905 b). In this work the author considers a measurable set, not necessarily reduced to an interval and not necessarily a subset of the real line, he calls *fundamental*: given a function on a fundamental set, he divides it in a finite or a countable number of components in an arbitrary manner and multiplies the content of every component for the least upper (resp. greatest lower) bound of the function in that component; he adds all these products and defines the *exterior* (resp. *interior*) *measure* of the integral as the greatest lower bound M (resp. least upper bound m) of all the sums relative to the upper (resp. lower) extremes. If these two measures coincide, then the function is called *integrable* in the new *generalized sense* and the *integral* is their common value.

This method of integration is very similar to Riemann's integration method if the fundamental set is an interval but in it countable partitions of the domain of the function constituted by Lebesgue measurable sets instead of intervals are considered. It is why in the second edition of the *Leçons* Lebesgue dwells upon Young's method and proves carefully the equivalence of the two formulations in the case the fundamental set is an interval and the function is bounded and Lebesgue measurable (Lebesgue 1950, 135).

In such a case, indeed, given the function $f(x)$ defined in the fundamental set E, Lebesgue considers for every sequence constituted by the following subsets of E:

$$e_n = E[n\varepsilon \leq f(x) < (n+1)\varepsilon],$$

the sums $s^L = \sum meas(e_n)n\varepsilon$ and $S^L = \sum meas(e_n)(n+1)\varepsilon$.

Consider now whatever partition of the fundamental set E constituted by the sets of whatever sequence (E_n); let δ_i be the measure of E_i and let l_i and L_i be respectively the greatest lower and the least upper bound of f in E_i: on the other hand, Young considers the sums of the series:

$$s=\sum l_i\,\delta_i; \quad S=\sum L_i\delta_i.$$

He considers the least upper bound, m, of the sums s and the greatest lower bound, M, of the sums S. Now the e_n are particular sets among those considered by Young: therefore, if m^L is the least upper bound of the sums s^L considered by Lebesgue and M^L is greatest lower bound of the sums S^L, we have:

$$m \leq m^L \leq M^L \leq M.$$

Therefore, every function that is Young summable is also Lebesgue summable. But vice versa, since it is possible to consider among the partitions made by Young those whose elements are all contained in one of the e_n then the difference M – m is less than $\varepsilon meas(E)$, therefore since every bounded and Lebesgue measurable function is Lebesgue integrable it is also Young summable and the integrals coincide.

Observe that the way to define the integral used by Young, differently from the Riemann's integration method can be applied also to unbounded functions: indeed, since the considered partitions are not necessarily finite, but can be countable, even if the function is unbounded around a point or a finite or countable number of points, it is possible to consider partitions constituted by a countable number of sets such that in every set of the partition the function is bounded.

For example, consider the function $f(x)$ defined in the following way: $f(x) = \frac{1}{\sqrt{x}}$ if $0 < x \leq 1$. Consider the partition of the interval (0, 1) constituted by the intervals $]\frac{1}{n+1}, \frac{1}{n}]$ with n belonging to the set N of natural numbers. Consider the *l.u.b.* of $f(x)$ in the *n-th* interval, $\sqrt{n+1}$, multiply it for the length of the *n-th* interval and consider then the infinite sum

$$\sum\left(\frac{1}{n} - \frac{1}{n+1}\right)\sqrt{n+1} = \sum \frac{1}{n\sqrt{n+1}}.$$

Since such a series is convergent, the exterior measure of the integral of $f(x)$ extended to the interval (0, 1) is finite. Analogously the interior measure of the integral of $f(x)$ is not less than $\sum \frac{1}{n\sqrt{n}}$. Therefore the function $f(x)$ is Young summable and is easy to verify that its integral is equal to 2.

Pierpoint's Definition of Integral

The previous definition by Young was generalized by James Pierpoint (1866-1938), a United States mathematician, who for many years was appointed as a full professor at Yale.

In the second volume of his *Theory of functions of real variable* (1912) Pierpoint considers firstly a new notion of partition of a set that may be also not measurable into sets that may be also not measurable. Precisely the set *E is divided into the separate sets* E_1 *and* E_2 if there exist two Lebesgue measurable sets M_1 and M_2 such that: $E_1 \subseteq M_1$, $E_2 \subseteq M_2$ and $meas(M_1 \cap M_2) = 0$.

If *E* is divided into the separate sets E_1 and E_2, then the following equality with respect to the exterior Lebesgue measure holds:

$$\overline{meas}(E) = \overline{meas}\,(E_1) + \overline{meas}\,(E_2)$$

and an analogous equality holds for the interior Lebesgue measure.

Pierpoint introduces also the notion of *measurability with respect to a set E* in the following way: given a not measurable set *E*, a set *F* is measurable with respect to *E* if there exists a Lebesgue measurable set *M* such that $F = E \cap M$.

Obviously every set is measurable with respect to itself. If *F* is measurable with respect to *E* then also *E-F* is measurable with respect to *E* and if *E* is Lebesgue measurable then a set is measurable with respect to *E* if and only if it is Lebesgue measurable.

It is possible to prove that *E* is divided into the two separate sets E_1 and E_2 if and only if they are measurable with respect to *E*.

Pierpoint also proves that:

If the set E is divided in a finite or countable number of sets E_i *all measurable with respect to E, or equivalently, constituting a separate division of E, then*

$$\overline{meas}(E) = \sum \overline{meas}\,(E_i). \qquad (*)$$

The same holds for the interior measure.

Once he has proposed these definitions, Pierpoint gives for the integral a definition like Young's one, but, instead of dividing the not necessarily measurable set *E* in a finite or countable number of measurable sets, he divides it in sets measurable with respect to *E* and considers their exterior measures.

The previous definitions caused a sharp criticism, expressed in two brief papers on the *Bull. Of the Amer. Math Soc.* by M. Fréchet, (Fréchet 1916) and (Fréchet 1917), who in those years wrote from the French front of the first world war, Pierpoint replied immediately (Pierpoint 1916) and (Pierpoint 1917). Substantially Fréchet had misunderstood the way to make the partitions. Indeed, if we consider traditional partitions, then in place of (*) we have the relation $\overline{meas}(E) \leq \sum \overline{meas}\,(E_i)$.

This entails the following difficulty: consider the function $f(x) = 1$ in the interval [0,1] and consider a partition constituted by the only interval [0,1] and another partition constituted by two not measurable sets A_1 and A_2 such that $\overline{meas}\,(A_1) + \overline{meas}\,(A_2) > 1$; then the upper integral of *f* extended to *E* = [0,1], that is equal to the g.l.b. of the sums $\sum \overline{meas}\,(E_i)$ where the partition (E_i) varies, is not greater than 1, while the lower integral which coincides with the l.u.b. of the same sums is not less than $\overline{meas}\,(A_1) + \overline{meas}\,(A_2) > 1$; therefore in this case the lower integral is strictly greater than the upper integral, a fact that damages the theory.

But, as we have seen, Pierpoint avoids such situations considering only particular partitions, precisely those which satisfy (*). Moreover, Fréchet observes that it is not essential to consider countable not finite partitions, but if his observation is perfectly pertinent for bounded functions, for unbounded functions countable partitions are necessary as we noticed exposing Young's integral.

Pierpoint answered Fréchet's criticism making clear the French mathematician had misunderstood his definitions and added:

> *To be historically accurate, I had no intention whatever of generalizing Lebesgue's integrals. When years ago I hit on my definition of integration, I did not know how it was related to Lebesgue's theory. I found out later that when the field of integration is measurable my integrals are identical with Lebesgue's and I have therefore called them Lebesgue integrals throughout my book. To prevent misunderstanding let me note that my definition is not restricted to a single variable …* (Pierpoint 1916).

However, Fréchet's criticism pointed out that Pierpoint's definition is not very useful for not measurable sets, while in the case of measurable sets it coincides with Young's one.

Young's Second Method of Integration

In 1911, after he had published a series of results about sequences of continuous and discontinuous functions, and almost as a completion of that study Young wrote a paper where he gave a functional definition of the integral for a larger class than the semicontinuous functions. Even if (Young 1911) passed over in silence almost completely, it would be in the sequel one of the main sources of inspiration for the generalization of integral carried out by Daniel in 1918.

Young begins from the class of the continuous functions on a bounded interval whose Cauchy integral is well known: as a first step, since if a bounded function is *lower semicontinuous* it is the limit of an increasing sequence of continuous functions, it is possible to define its integral as the limit of the integrals of such functions; analogously if the function is *upper semicontinuous*, since in such a case it is the limit of a decreasing sequence of continuous functions; the second step consists in giving the definition of *lower generalized integral* of whatever bounded function *f(x)* as the least upper bound of the integrals of the upper semicontinuous functions not greater than *f(x)* and of *upper generalized integral* of *f(x)* as the greatest lower bound of the integrals of the lower semicontinuous functions not less than *f(x)*. Young proves that:

> *it is possible to determine an increasing (decreasing) monotone sequence of upper (lower) semicontinuous functions not greater than f(x) (not less than f(x)) whose integrals have the lower (upper) generalized integral as their limit* (Young 1911, 27).

In this way upper and lower generalized integrals are not more looked on as lower and upper extremes of integral sums in the manner of Darboux, but they are lower and upper extremes of integrals.

A function is called *summable* if the upper and lower generalized integrals coincide; the preceding definition is then extended also to unbounded functions.

Young proves that obviously every bounded and Lebesgue measurable function is summable in the new sense. Notice that if a function is Lebesgue summable then Young's condition is a theorem, known as *Vitali-Caratheodory Theorem*.

Moreover he proves that:

> *every bounded summable function coincides almost everywhere with the least upper bound of a sequence of upper semicontinuous functions and with the greatest lower bound of a sequence of lower semicontinuous functions* (Young 1911, 16).

Thus he concludes that every summable function can be modified on a null measure set in such a way to be included in Baire's first class of functions. Indeed he proves, denoting by *lu* (*ul*) a function that is least upper bound (greatest lower bound) of an increasing (decreasing) sequence of upper (lower) semicontinuous functions:

> *a function belongs to the Baire's first class if and only if it is both lu and ul* (Young 1911, 24).

Following Baire, Young iterates the second step, alternating limit of increasing sequences with limit of decreasing sequences. In this way he defines the integral for every Baire function of the first class.

What Class Do Measurable Functions Belong To?

Baire functions were introduced by Baire in his already quoted doctorate thesis in 1899; they are defined by steps: at step 0 there are the continuous functions; at step 1 the bounded or unbounded discontinuous functions that are punctual limits of continuous functions; at step 2 the discontinuous functions that do not belong to class 1 but are punctual limits of sequences of functions of class 1 and so on functions of every finite order are defined; now it is possible to imagine that a function is a limit of functions of finite increasing orders but does not belong to anyone of the previous classes. Then this function is said to belong to the first transfinite class ω. If a function is the limit of a sequence of functions of class ω is called of class $\omega + 1$ and so on it is possible to define functions of every transfinite order.

The subject was widely discussed at the beginning of the twentieth century and studied from different points of view. Immediately some mathematicians observed that all the Baire's classes are constituted by measurable functions. For example, in June 1905 Guido Fubini wrote in a post card addressed to Vitali:

> *all the Baire's functions are measurable being limits of measurable functions. Therefore they are the sum of a function of second class and of a function whose integral is zero … Therefore it is useless to study functions of class 3, 4 and so on …* (Vitali 1984, 454).

Also Lebesgue had written in 1902, in his thesis, that all the Baire's functions are summable (but at that time he still used the world summable in place of measurable); in 1904 he affirms that the Baire's functions are all B measurable (Lebesgue 1904a, 111-112). In other papers Lebesgue often considered Baire's functions, for example (Lebesgue 1904b) is devoted to an important analysis of functions of the first class.

Also Vitali dealt with this topic in a brief note (Vitali 1905a) where he proves that:

> *A real measurable function defined in an interval (a, b) is the sum of a function of class not greater than 2 and a function which is zero everywhere except on a null measure set.*

This note is worthy of great attention since in it Vitali enunciates and proves the theorem that will become famous under the name of Lusin's Theorem. Vitali alludes in a doubting form to a possible coincidence with statements by Borel and Lebesgue that appeared on the C. R. in 1903, without demonstration and that were taken up in the *Leçons* in 1904, where in a note Lebesgue writes:

> *Every measurable function is continuous except on the points of a null measure set, when one neglects the sets of measure ε, whatever ε is* (Lebesgue 1904a, 125, note 1).[11]

[11] Toute fonction mesurable est continue, sauf aux points d'un ensemble de mesure nulle, quand on néglige les ensembles de mesure ε, si petit que soit ε.

Lebesgue will remember his priority in the formulation (even if incomplete) of this fundamental property in the letter of 1907 February 16 (Vitali 1984, 457).

Again in 1905, in a brief note, Vitali demonstrates what Fubini had affirmed in the postcard not long before:

> *A measurable function can be decomposed in the sum of a function of the second class and a function with null integral* (Vitali 1905c).

Vitali considers again the problem where he proves the reverse proposition of a proposition from Lebesgue's *Leçons*:

> *every B measurable function is a Baire function* (Vitali 1905b).

The paper is also interesting since in it Vitali introduces the notion of *truncated function.*

A deep study of the Baire's classes of finite or transfinite order was achieved by de La Vallée Poussin in his treatise of 1916 (de La Vallée 1916); and in (de La Vallée 1915), he operates even a detailed classification of the B measurable sets, following (Borel 1912). Indeed, afterwards the complete equivalence between belonging to a Baire's class and being a B measurable function was made clear both Borel and de La Vallée Poussin intended to establish a sort of hierarchy for the constructive functions and sets. In the Preface of (de La Vallée 1916) the Author underlines that these notions define a completely sufficient sphere of activity for the mathematician, beyond which every generalization cannot induce more real contributions, but has only a philosophic or an aesthetic meaning.

Let us come back to see how Young proves that the bounded measurable functions are a. e. least upper bounds of upper semicontinuous functions and greatest lower bounds of lower semicontinuous functions and concludes that:

> *every bounded measurable function coincides a. e. with a Baire function of the first class* (Young 1911).

Analyse the proof by Young: he remembers his result that the limit of a decreasing (increasing) sequence of upper (lower)semicontinuous functions in a point P is upper (lower) semicontinuous in P. Now, he asserts, a semicontinuous function defined in an interval I belongs to the first class, then, for a theorem by Baire, it is continuous with respect to I in the points of a dense subset of I. Then, given an increasing sequence of upper semicontinuous functions, in every interval there are points where the functions are continuous and therefore lower semicontinuous (but all the functions in the same points? no word is spent about this problem); therefore, the limit function is lower continuous. Reasoning in this way Young concludes that every function that is both *ul* and *lu* is continuous in all the points of a dense perfect subset of I and therefore belongs to the Baire first class. He does not explain why, given a sequence of upper semicontinuous functions (and therefore Baire functions of the first class) there exists a dense subset where all the functions are contemporaneously continuous: this is a crucial point in his demonstration, its prove is present in the literature of those years and is given in the following section.

Pointwise Discontinuous Functions

The German mathematician Hermann Hankel (1839-1873) first introduced a new notion about discontinuous functions in an important memory presented by him to the University of

Tubingen in 1870 and published after his untimely death in 1873. In such a memory he proved that:

> *if a function f is Riemann integrable in an interval I then for every σ>0 the set of the points x such that the oscillation of f in x is greater than σ is a nowhere dense subset of I.*

Hankel believed incorrectly that the previous condition was also a sufficient condition in order that a function is Riemann integrable, but we are not interested in such a question about which see for example (Letta 1994). Instead let us consider now the following Hankel's theorem:

> *if for every σ>0 the set of the points x such that the oscillation of f in x is greater than σ is a nowhere dense subset of I then the set of the points where f is continuous is a dense subset of the interval I.*

Indeed, let us suppose that, for every σ>0, the set D_σ constituted by the points where the oscillation is greater than σ is nowhere dense; then given whatever interval U there exists a closed interval $[a_1, b_1] \subseteq U$ where there are not points from D_σ; for the same reason there exists a closed interval $[a_2, b_2] \subseteq [a_1, b_1]$ where there are not points from $D_{\frac{\sigma}{2}}$. Iterating this procedure it is possible to construct a decreasing sequence of intervals $\{[a_n, b_n]\}$ such that in $[a_n, b_n]$ there are not points from $D_{\frac{\sigma}{2^n}}$. All these intervals have at least one point in common which is a point where the function is continuous. Then in every interval U there is a point where the function is continuous.

The discontinuous functions such that the set of the continuity points is a dense subset of I, are called by Hankel *punktirt unstetige Functionen* in contrast with the *total unstetige Functionen*. They are studied by Ulisse Dini (1845-1918) in his treatise (Dini 1878), where he calls them in Italian *punteggiate discontinue*. In the English literature they are known as *pointwise discontinuous functions*. Vito Volterra (1860-1940), Dini's disciple, wrote in 1881 a paper about them, where, using skilfully Hankel's technique, among other things, he proves that:

> *If a pointwise discontinuous function defined in an interval I has discontinuity points in every subinterval of I, then there is no pointwise discontinuous function that is continuous in the points where the first function is discontinuous and is discontinuous where it is continuous* (Volterra 1881, 76-77).

Indeed, let *f* and g be two pointwise discontinuous functions defined in the interval (a, b). Let (α, β) a subinterval of (a, b) and let $\varepsilon > 0$; since *f* is continuous at least in a point of (α, β), there is a subinterval of (α, β) where the oscillation of *f* is less than ε and inside this interval for the same reason there is another closed interval (a_1, b_1) where the oscillation of g is less than ε too. This reasoning can be repeated starting from the interval (a_1, b_1) and considering $\frac{\epsilon}{2}$ in place of ε; it is possible to determine a closed interval $(a_2, b_2) \subseteq (a_1, b_1)$ where both the oscillations of *f* and g are less than $\frac{\epsilon}{2}$. iterating the procedure it is possible to determine a sequence of closed intervals $\{(a_n, b_n)\}$ such that in the interval (a_n, b_n) both the oscillations of *f* and g are less than $\frac{\epsilon}{2^n}$. All the intervals of the sequence have a common point, *M*, where both the oscillations of *f* and g are zero, therefore both *f* and g are continuous in *M*.

A corollary of the previous proposition is the following:

Given a finite number of pointwise discontinuous functions defined in the same interval I, there exists a dense subset of I where all the functions are continuous contemporaneously.

The following theorem given in 1904 by Lebesgue (Lebesgue 1904 b, 236) generalizes Volterra's result to the case of infinitely many functions; it is proved repeating substantially Volterra's demonstration:

Given a sequence of Baire functions of the first class, all defined in the same domain, there exists a dense subset of the domain where all the functions are continuous contemporaneously.

Observe that, as we have already seen, Baire had proved the following theorem:

A function belongs to the first class if and only if it is pointwise discontinuous.

Therefore, the Baire's first class functions which Lebesgue is concerned to are the pointwise discontinuous functions used by Volterra.

We can at this point conclude completely Young's argumentation: if the function f is both *ul* and *lu* then there exist two sequences, one increasing of upper semicontinuous functions, the other decreasing of lower semicontinuous functions, whose limit is f. Now by the just quoted theorem by Lebesgue, there exists a dense subset where all the functions of the two sequences are continuous: therefore, in such points f is both lower and upper semicontinuous, and therefore continuous. Then it is pointwise discontinuous and belongs to the Baire's first class.

The Definition of Integral by Borel and Hahn

A new definition of integral originates from an idea by (Borel 1912); with a polemic purpose towards Lebesgue, Borel resumes Riemann's definition and its generalizations by Jordan and Harnack and avoids the recourse to Lebesgue's measure theory considered not much natural, not much direct and not much constructive. Indeed, Borel's definition of a measurable set takes into account its composition out of intervals, a quality which Borel found constructive and therefore philosophically attractive, but, as we will see, it is not completely adequate in order to build on it a general theory of integral, see also (Hawkins 2002, 105). In the same paper Borel gives the definitions of *computable number* and *computable function* he extensively comments since he thinks that only constructive methods can be useful for mathematics.

Afterwards, in 1915, the new definition and the correlated theory were improved and emended from some of their imperfections by the Austrian mathematician Hans Hahn (1879-1934). To improve the new theory, he proposed to extend Riemann's integral to bounded functions defined in a closed set *H* contained in a closed and bounded interval $[a, b]$ (Hahn 1915). He observed that this can be obtained easily in the following way: consider a partition of the interval $[a, b]$ obtained by inserting the points $x_0 = a < x_1 < \dots < x_n = b$: in the sum $\sum_{i=1}^{n} f(\varphi_i)(x_i - x_{i-1})$ in place of the length $(x_i - x_{i-1})$ put the content[12] of the set $H\cap$ [xi-1, xi] $H\cap[x_{i-1}, x_i]$ and chose whatever point φ_i in such a set. Continue in the usual way.

Borel furnished the following:

[12] The content in general coincides with the exterior Peano-jordan's measure. But the set we are considering is a closed set and in this case the content coincides with the Lebesgue's measure.

Definition – Let f(x) be a not bounded function, continuous in the interval [a, b] except in the points of an enumerable set Z, contained for every ε > 0 in the union A_ε of a sequence of open intervals each of them containing at least a point of Z and whose total length is less than ε. Then the Riemann's integral exists on the complement H_ε of A_ε that is a closed set. If there exists finite the $\lim_{\varepsilon\to 0}\int_{H_\varepsilon} f(x)dx$ independent from the choice of the intervals A_ε, the function f(x) is called Borel integrable and the previous limit is by definition the Borel integral of f(x) on [a,b], denoted by (B)$\int_a^b f(x)dx$.

Obviously if the absolute value of a function is Borel integrable then also the function is Borel integrable. Borel considers only unbounded functions but his reasoning can be used also for bounded functions: it is evident that in such a case we do not obtain more than the Riemann's integral. Hahn observed that if a function is absolutely Borel integrable then it is also Lebesgue integrable and the two integrals coincide; indeed by the preceding definition if *f* is nonnegative and $\varepsilon = \frac{1}{n}$ then we have, considering an increasing sequence of closed sets $H_n \subseteq [a, b]$ such that $m(H_n) > b - a - \frac{1}{n}$ for every *n*:

$$(B)\int_a^b f(x)dx = \lim_n (B)\int_{H_n} f(x)dx = \lim_n (L)\int_{H_n} f(x)dx = (L)\int_a^b f(x)dx,$$

where Beppo Levi's monotone converge theorem has been applied.

Notice that if a function is Lebesgue integrable then it is also absolutely integrable, but there exist Borel integrable functions that are not absolutely integrable as for example the function $f_1(x) = \frac{1}{x}\sin\frac{1}{x}$. Moreover there exist nonnegative functions which are Lebesgue integrable but not Borel integrable as for example the characteristic function of $[0, 1] \cap Q$, where *Q* is the rational numbers set: indeed this function is Lebesgue integrable, but totally discontinuous and therefore not Borel integrable.

Borel in the second edition of his *Leçons sur la théorie des fonctions* (Borel 1914) highlights a fault of his definition: the sum of two Borel integrable functions is not necessarily Borel integrable since *the intervals of exclusion of one of the two functions must be chosen in some particular case in such a way to make divergent the integral of the other function*. For example consider the functions in the interval [0,1]:

$$f_1(x) = \frac{1}{x}\sin\frac{1}{x}; \qquad f_2(x) = \sum_{n=1}^{\infty} \frac{e^{-n}}{\sqrt{|2-(2n+1)x|}}.$$

While $f_1(x)$ has only one singular point, the point $x = 0$, the function $f_1(x) + f_2(x)$ has obviously the singular point $x = 0$ but has also all the singularities of $f_2(x)$, that is all the points $x = \frac{2}{2n+1}$ for $n = 1, 2, \dots$. Both the considered functions are Borel integrable but for their sum it is not possible to apply previous Borel's definition (Borel 1914). Borel's example is somewhat imprecise and will be emended and clarified in 1915 by Hahn, whose reasoning is exposed in this paper in the sequel.

Besides, Borel's definition of (B)-integral presents two excessively restrictive conditions:

a) the singularities have to constitute a countable set: it is evident that this condition can be easily removed and can be substituted by the hypothesis that the set of the singularities is a zero measure set;

b) In Borel's definition every open interval that constitutes the set A_ε to be excluded has to contain at least one singular point and as we will see in detail in the following pages this can involve difficulties when considering the sum of two Borel integrable functions.

For these reasons Hahn resumed successively Borel's formulation and changed it in some points to avoid the previous difficulties (Hahn 1915). Here is his definition, where he uses the Riemann's integral extended to a closed set we have exposed before.

Let f(x) be a function defined on the interval [a, b] such that:

a) *For every $\varepsilon > 0$ there exists a perfect set H_ε whose measure is greater than a-b-ε and such that the Riemann's integral $\int_{H_\varepsilon} f(x)dx$ exists;*

b) *The limit $\lim_{\varepsilon\to 0}\int_{H_\varepsilon} f(x)dx$ exists, whatever the system of sets H_ε satisfying previous conditions is.*

Then such a limit is called (B)-integral of f on [a, b]. If the previous limit is finite then the function is (B*)-integrable.*

Obviously the previous definition holds also for the integral extended to a measurable set *E* different from [*a, b*]. Observe that condition a) entails, by Vitali- Lebesgue's Theorem that the function *f(x)* is almost everywhere continuous in the perfect set H_ε and therefore, by Vitali–Lusin's Theorem, this condition implies that *f(x)* is Lebesgue measurable.

Hahn proves that:

if f is bounded in [a, b] and condition a) holds then the (B)-integral of f on [a, b] exists and is finite, that is the function f is (B*)-integrable.*

Moreover:

if f is Lebesgue integrable then it is also (B)-integrable.*

Indeed, let *f* be measurable on [*a, b*]: by Vitali-Lusin's Theorem for every natural *n* it is possible to modify *f* on a set whose measure is less than $\frac{1}{2^n}$ in order to obtain a continuous function f_n on [*a, b*]. It is possible to prove that the sequence (f_n) converges almost everywhere on [*a, b*]. Then by Severini-Egoroff's Theorem for every *n* it is possible to extract a subsequence (f_{n_k}) that is uniformly convergent to *f* in a closed set H_n such that $m([a, b]-H_n)<\frac{1}{n}$; since *f* is continuous on H_n it is then possible to consider the integral $\int_{H_n} f(x)dx$; since $m([a, b]-\cup H_n)=0$ and *f* is Lebesgue integrable, the Lebesgue integral of *f* extended to [*a, b*] coincides with $\lim_n \int_{H_n} f(x)dx$. Then this limit exists and is finite, that is *f* is (B*)-integrable.

If *f* is (B*)-absolutely integrable then it is immediately seen that, as in the case of a (B)-absolutely integrable function, *f* is also Lebesgue integrable. But now what is very interesting is the fact that if *f* is (B*)-integrable then it is also Lebesgue integrable, that is:

f is (B)-integrable if and only if f is Lebesgue integrable.*

Indeed:

If f is (B)-integrable then it is also (B*)-absolutely integrable.*

The demonstration of this implication given here is deduced from that of the original paper (Hahn 1915) and from the version given in the memoir (Hildebrand 1917), with some little changes to make it more understandable.

Let f be (B*)-integrable, but not (B*)-absolutely integrable. Let $f^+ = \frac{1}{2}(|f| + f)$, $f^- = \frac{1}{2}(|f| - f)$. Then, since (B*)$\int(f^+ - f^-) < +\infty$ and (B*)$\int(f^+ + f^-) = +\infty$, it has to be (B*)$\int f^+ = +\infty$ and (B*)$\int f^- = +\infty$.

By condition a) for every ε>0 a perfect set P_ε exists such that m[13]$(P_\varepsilon) > b\text{-}a\text{-}\varepsilon$ and f^- is Riemann integrable and therefore bounded on P_ε, let $f^- < m_\varepsilon$ on P_ε. Then $\int_{P_\varepsilon} f^- < m_\varepsilon(b - a)$. Since (B*)$\int f^+ = +\infty$ for every $M > 0$ and $\varepsilon > 0$ there exists a positive number r' such that for every $r < r'$ a perfect set P_r exists such that $m(P_r) > b\text{-}a\text{-}r$ and $\int_{P_r} f^+ > M + m_\varepsilon(b - a)$. It is possible to suppose $P_\varepsilon \subseteq P_r$.
Let $E_1 = \{x \in P_\varepsilon: f(x) < 0\}$; $E_2 = \{x \in P_r: f(x) \geq 0\}$: these sets are measurable and disjoint and their union E is measurable, $P_\varepsilon \subseteq E \subseteq P_r$ and therefore $m(E) > b\text{-}a\text{-}\varepsilon$. Moreover:

$$(L)\int_E f = \int_{P_r} f^+ - \int_{P_\varepsilon} f^- > M.$$

Then for every $M > 0$ and $\varepsilon > 0$ a measurable set E exists such that $m(E) > b\text{-}a\text{-}\varepsilon$ and (L)$\int_E f > M$, f being Lebesgue integrable on E.
Now, by Lebesgue's Theorem, for every increasing sequence of closed sets enclosed in E such that $m(E) = lim_{n\to\infty} m(P_n)$ it is:

$$(L)\int_E f = lim_{n\to\infty}(L)\int_{P_n} f = lim_{n\to\infty}(R)\int_{P_n} f.$$

And therefore, since $(L)\int_E f > M$ there exists a natural number μ such that for every $n > \mu$ it is $(R)\int_{P_n} f > M$.
It is possible to choose n such that $m(P_n) > b - a - 2\varepsilon$. Then for every $\varepsilon > 0$ and for every $M > 0$ there exists a closed set H ($=P_n$) $\subseteq [a, b]$ such that $m(H) > b\text{-}a\text{-}2\varepsilon$ and $(R)\int_H f > M$, that is f is not (B*)-integrable, contrarily to our hypothesis.
There exist (B)-integrable functions that are not (B*)-integrable, for example the function $f_1(x) = \frac{1}{x}\sin\frac{1}{x}$. Such a function is not absolutely (B)-integrable, therefore it is not Lebesgue integrable, thus it cannot be (B*)-integrable. It is possible to prove directly that f_1 is not (B*)-integrable in the following way. Indeed, given $\varepsilon > 0$ let u be an even natural number such that $\frac{1}{u} < \varepsilon$ and let k be whatever natural number. Consider:

$$P_\varepsilon = [\frac{1}{ku+1}, \frac{1}{ku}] \cup \ldots \cup \left[\frac{1}{u+1}, \frac{1}{u}\right] \cup [\frac{1}{u}, 1];$$

here we suppose that, except the last one, only and all those intervals whose second extreme is the reciprocal of an even number are considered. The function f_1 is positive in these intervals, except in the last one. It is $m(P_\varepsilon) > 1\text{-}\varepsilon$.
Moreover:

$$\int_{P_\varepsilon} f_1 = \int_{\frac{1}{ku+1}}^{\frac{1}{ku}} \frac{1}{x} sen\frac{\pi}{x} dx + \ldots + \int_{\frac{1}{u+1}}^{\frac{1}{u}} \frac{1}{x} sen\frac{\pi}{x} dx + \int_{\frac{1}{u}}^{1} \frac{1}{x} sen\frac{\pi}{x} dx =$$

$$= \int_{ku}^{ku+1} \frac{1}{t} sen\pi t dt + \ldots + \int_{u}^{u+1} \frac{1}{t} sen\pi t dt + \int_{\frac{1}{u}}^{1} \frac{1}{x} sen\frac{\pi}{x} dx$$

$$\geq \frac{2}{\pi}[\frac{1}{ku+1} + \frac{1}{ku-1} + \ldots + \frac{1}{u+1}] + \int_{\frac{1}{u}}^{1} \frac{1}{x} sen\frac{\pi}{x} dx$$

[13] m is the Lebesgue measure.

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$\geq \frac{2u}{2\pi}[\frac{1}{ku+1} + \frac{1}{(k-1)u+1} + \ldots + \frac{1}{3u+1} + \frac{1}{2u+1}] + \int_{\frac{1}{u}}^{1} \frac{1}{x} sen \frac{\pi}{x} dx.$

Then we have: $lim_{\varepsilon \to 0} \int_{P_\varepsilon} f_1 \geq \frac{1}{\pi}[\frac{1}{k} + \ldots + \frac{1}{3} + \frac{1}{2}] + (B)\int_0^1 \frac{1}{x} sen \frac{\pi}{x} dx,$

whence, since k is arbitrary, we deduce that the limit on the right is not finite and therefore f_1 is not (B*)-integrable. This situation is linked to the nature of the (B*)-integral: in the case of the (B)-integral the sets Pε have to exclude only the singular point x = 0 and no other intervals that do not contain singularities. Instead in the case of the (B*)-integral we have included in P_ε the intervals where the function is positive that do not contain singularities of the function and they are not automatically compensated by the intervals where the function is negative as in the case of the (B)-integral.

Observe also that the sets Pε considered above exclude a neighbourhood of the origin and some open intervals that are neighbourhoods of the points x=$\frac{2}{2n+1}$ where *n* is an odd natural number. By means of the previous argument it is then possible to state precisely the not completely clear example given by Borel about the sum of two (B)-integrable functions. Consider the function f_1 and the function f_2' obtained modifying the function f_2 proposed by Borel in such a way that the sum is extended only to the odd natural numbers *n*. Then the previous argument proves that the function $f_1 + f_2'$ is not (B)-integrable.

Lebesgue Stieltjes Integral

The Dutch mathematician Thomas Johannes Stieltjes (1856–1894) was not motivated for his definition of integral by the purpose to give a generalization of Cauchy's theory of integral but by his own study of continuous fractions. On this subject in his interesting memory *Recherches sur les fractiones continues*, published on the Comptes Rendus de l'Académie des Sciences, he had introduced ever since 1894 the concept of distribution of mass, much utilized in physics but in that time unusual in mathematics.

Suppose that a distribution of *n* positive masses m_i is given on the positive *x*-axis, each placed at a distance d_i from the origin. The moment of order *k* of the system is then defined by $\sum_{i=1}^{n} m_i d_i^k$. Stieltjes introduced the notion of a continuous distribution of mass: it is completely determined when $\alpha(x)$, the total mass between 0 and *x*, is given for every positive *x*. Obviously the function $\alpha(x)$ must be positive and not decreasing. The moment of order *k* in this case is defined as the limit of the sums $\sum_{i=1}^{n} t_i^k[\alpha(x_i) - \alpha(x_{i-1})]$, where $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ is a partition of the interval $[a, b]$, t_i is any point of $[x_{i-1}, x_i]$ and the maximum length of the intervals of the partition tends to 0. In general, if *f*(t) is any continuous function the previous sum becomes:

$$\sum_{i=1}^{n} f(t_i)\,[\alpha(x_i) - \alpha(x_{i-1})]$$

and it is possible to see that it tends to a limit.

Stieltjes does not go any further, but, obviously, it is possible to extend the previous definition to the case α is a function of bounded variation. In such a way, as for the Lebesgue integral, a functional arises associating with every continuous function its Stieltjes integral. In the first years of the twentieth century some papers of great interest about the integral representation of functionals appear.

In 1909 Riesz proves that for every linear and continuous functional *A* on C(*a*, *b*), the class of the continuous functions in (*a*, *b*) defined in (*a*, *b*) and equipped with the norm of the maximum, there is a function α of bounded variation such that for every $f \in$ C(*a*, *b*) the functional is expressed by the Stieltjes integral:

$$A(f) = \int_a^b f(x)d\alpha(x).$$

As Riesz himself writes in 1949, in a paper of historical character, also Hadamard had dealt with this problem in 1903: in those times he had proved that for every such a functional *A* there exists a sequence of Riemann integrable functions $\{a_n\}$ such that $A(x) = lim_n \int a_n(t)x(t)dt$ (Riesz 1949). In 1909 Riesz improves this result and determines the previous expression of the functional as a Stieltjes integral.

A year later Riesz establishes the integral representation of the linear and continuous functionals in the space L_p (Riesz 1910).

Lebesgue was much interested in this result, and in the second edition of his *Leçons sur l'intègration*, in the chapter XI, extends the notion of Stieltjes integral to a larger class of functions (Lebesgue 1950, Preface of the second edition 1926 December 3). He introduces a measure correspondingly to any function not decreasing and nonnegative $\alpha(x)$ in the following way. For any interval $[a, b]$ Lebesgue sets: $\alpha([a, b]) = \alpha(b+)-\alpha(a-) = \lim_{x\to b^+} \alpha(x) - \lim_{x\to a-} \alpha(x)$, where the function and the arising set function are denoted by the same symbol; then for every set *E* he observes that it is possible to construct interior and exterior measures of *E* starting from the definition on the intervals and therefore it is possible to define the corresponding measurable sets in Lebesgue style. Proceeding in the construction further it is possible to see that every measurable set obtained in this manner is a Lebesgue measurable set. In such a way Lebesgue obtains a completely additive measure α, defined on the Lebesgue measurable sets. He then defines the prolongation of the measure α to the measurable sets and, in order to define the Lebesgue Stieltjes integral for a measurable function, he proceeds as in the case of the Lebesgue integral but in this case, he values the sets by the prolongation of the measure α instead of valuing them by the Lebesgue measure (Lebesgue 1950, 277-281). The new integral is denoted by 21

$$\int_a^b f(x)d\alpha(x).$$

In the case the function is whatever function of bounded variation it can be written, by a Jordan's theorem, as the difference *p-q*, where *p* and *q* are nonnegative and not decreasing. Therefore if α is a function of bounded variation then $\alpha = p-q$ and the Stieltjes integral with respect to α is defined in the following way:

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)dp(x) - \int_a^b f(x)dq(x).$$

Lebesgue underlines that for the new integral $\int_a^b f(x)d\alpha(x)$ all the properties of the classical integrals can be proved directly if we observe that it is possible to give it also another form. To this purpose if α(x) is a continuous and strictly increasing function it admits an inverse function of the same kind, $x = x(\alpha)$ and the previous integral can be written as an usual integral in the form:

$$\int_a^b f(x)d\alpha(x) = \int_{\alpha(a)}^{\alpha(b)} f[x(\alpha)]d\alpha.$$

The integral on the right does make sense even if *f* is not continuous but is measurable and bounded or in the case *f* is not bounded but $[f[x(\alpha)]$ is summable. An analogous relation can be written even if $\alpha(x)$ has some jumps or is constant in some interval. In the case $\alpha(x)$ is of bounded variation, as we have already seen, it is possible to express it as the difference of two increasing functions and to refer to the previous case.

The Generalizations of the Lebesgue Stieltjes integral by Radon and Fréchet

About twenty years after the memory by Stieltjes some authors introduced approaches more abstract for the notions of measure and integral: in this scheme of things Johann Radon (1887-1956) carried out a general study (1913) where the definitions of integral by Stieltjes and Lebesgue are still included and referred to the *n*-dimensional Euclidean space and Lebesgue's measure. Radon considers integrals of the type:

$$\int F(P)d\alpha(P)$$

where the integrand is a function *F*(*P*) of the point *P* and the integral is referred to a function $\alpha(P)$ of bounded variation: since he still considers Lebesgue measurable sets, his definition coincides with Lebesgue's one if the function α is linear and with the definition by Stieltjes if *F* is continuous.

Fréchet completes the process of abstraction two years after, removing the restriction to the *n*-dimensional space (Fréchet 1915). He takes Radon's definition as a starting point and observes that the great advantage of the extension of the integral obtained by him, perhaps escaped even to Radon, consists in the fact that it is possible to write it in the form:

$$\int F(P)df(e)$$

where the integral is extended to a subset *E* of the real or *n*-dimensional space and *f(e)* is an additive function of the varying subset *e* of *E*.

Fréchet thinks that it is no more necessary to refer to the Lebesgue measurable subsets of an Euclidean space, but it is possible to define integrals of functions (that Fréchet calls *functionals*) defined in an abstract set *E* . He supposes that a family *F* of subsets of *E* is given on *E*, he calls an *additive family of sets* and today is called a σ-algebra of sets. In place of a function of bounded variation α or the Lebesgue measure he introduces any additive function of sets, today called a relative measure, that is a real function defined on *F* such that given any sequence $\{A_n\}$ of elements of *F*, pairwise disjoint, whose union is *A*, it is: $f(A) = \sum f(A_n)$, and defines the integral of any function defined in *E* with respect to this relative measure in a rather natural manner. In such a way it is possible to obtain a theory wider than the theory of Lebesgue's integral and the theory of Stieltjes integral, that is a generalized Radon theory.

Let us see the first steps of his construction: Fréchet proves that as any function of bounded variation can be written as the difference of two nonnegative non decreasing functions, by the Jordan's decomposition, also the relative measure *f* can be decomposed in the difference of two additive set functions, γ and δ, both nonnegative. Then, given a bounded function *F* defined in *E*, the integral of *F* extended to *E* is defined in the following way: consider a partition $\{E_n\}$ of *E* constituted by a finite number of elements of *F* and form the sums

$$S' = \sum M_n\gamma(E_n), \qquad s' = \sum m_n\,\gamma(E_n),$$

where M_n and m_n are the least upper bound and the greatest lower bound respectively of *F* in E_n. As the partition varies *S'* has a finite greatest lower bound that is called *upper integral of F in E* with respect to γ and the family *F*. It is denoted by $\overline{\int_E} F(P)d\gamma(e)$.

In the same way *s'* has a finite least upper bound that is called *lower integral of F in E* with respect to γ and the family *F*. It is denoted by $\underline{\int}_E F(P)d\gamma(e)$.

Fréchet proceeds in an analogous way with the additive function δ. Then by definition the *upper integral of F on E with respect to f and the family F* is the following finite number:

$$\overline{\int}_E F(P)df(e) = \overline{\int}_E F(P)d\gamma(e) - \underline{\int}_E F(P)d\delta(e)$$

The *lower integral of F on E with respect to f and the family F* is defined analogously by:

$$\underline{\int}_E F(P)df(e) = \underline{\int}_E F(P)d\gamma(e) - \overline{\int}_E F(P)d\delta(e).$$

As for the integral of Darboux it is possible to prove that the lower integral of F on E with respect to a measure *f* is not greater than the upper integral.

If the upper integral of F on E coincides with the lower integral, then the function *F* is called *integrable on E* and its *integral* is by definition the common value of the upper and lower integrals: it is denoted by

$$\int_E F(P)df(e).$$

In such a case the function *F* is also integrable with respect to γ and δ and, analogously to the decomposition of the Stieltjes integral, it is:

$$\int_E F(P)df(e) = \int_E F(P)d\gamma(e) - \int_E F(P)d\delta(e).$$

Fréchet deals also with the case of not bounded functions, but the preceding exposition can be sufficient to understand how he proceeds in his construction.

Lebesgue Stieltjes Integral by Caccioppoli

We expose now the original contribution to the definition of Stieltjes-Lebesgue integral in more dimensions given by the young Napolitan mathematician Renato Caccioppoli (1904-1959) in one of his first papers [Caccioppoli 1926 a]; there he extends Riesz theorem about the dual space of C([0, 1]), in the simplified version published in 1914, to the case of functions of more variables and the definition of the functions of bounded variation to the case of more variables in accordance with Vitali's pioneering work (Vitali 1908) and a paper by Fréchet (Fréchet 1910). To this end he considers a real bounded function $f(x, y)$ defined in a set *I* enclosed in a rectangle R whose edges are parallel to the axes *x*, *y*. He subdivides R in a finite number of partial rectangles by means of the right lines of equation $x = x_i$, $y = y_j$ i = 1, 2, … *n*, j = 1, 2, … *m* and considers the rectangle R_{ij} defined by the inequalities: $x_i \leq x \leq x_{i+1}$, $y_j \leq y \leq y_{j+1}$. It is also given in R a function $u(x, y)$, whose double variation in the rectangle R_{ij} is given by:

$$\Delta_i \Delta_j u = u(x_{i+1}, y_{j+1}) - u(x_{i+1}, y_j) - u(x_i, y_{j+1}) + u(x_i, y_j).$$

The function $u(x, y)$ is of (*double*) *bounded variation* if $\sum_{i,j} |\Delta_i \Delta_j u|$ is upper bounded when the partition $\{R_{ij}\}$ varies and, if this is the case, the total variation of $u(x, y)$ is the l.u.b of $\sum_{i,j} |\Delta_i \Delta_j u|$.

Let $f_{i,j}$ be a value between the extremes of $f(x, y)$ in $R_{i,j}$ and form the sum: $\sum_{i,j} f_{i,j} \Delta_i \Delta_j u$, where the sum is extended only to those $R_{i,j}$ such that $R_{i,j} \cap I$ is not empty. If this sum tends to a finite limit when the maximum diameter of the $R_{i,j}$ tends to zero, this limit is called the *Stieltjes integral* of $f(x, y)$ with respect to the function $u(x, y)$ extended to *I*.

It is easily proven that if $f(x, y)$ is continuous and $u(x, y)$ is of bounded variation then the Stieltjes integral exists finite.

At this point Caccioppoli proves that given a linear continuous functional A in the set of the continuous functions in a closed and bounded plane set I, $C(I)$, there exists a function of bounded variation $u(x, y)$ such that for every function f(x,y) continuous in I

$$A[f(x, y)] = \iint_I f(x, y)d_x d_y u(x, y) \qquad (*)$$

where the Stieltjes integral of f(x,y) with respect to u extended to I appears on the right.

The proof is as follows: the Author considers the set $C(R)$ of the continuous functions in a rectangle R and a linear and continuous functional $A(f)$ defined in $C(R)$, that is such that if $f_n \in C(R)$ tends uniformly to f then $A(f_n)$ tends to $A(f)$. Such a functional is also bounded that is there exists a constant K such that $A(f) \leq K \max |f(x, y)|$. It is possible to prove that if $f_n \in C(R)$ constitute an increasing sequence that tends pointwise to a bounded function f then $A(f_n)$ tends to a finite limit, that depends only on f, that is it is the same for every increasing sequence tending to f.

Now Caccioppoli considers a rectangle $R(O, Q)$ whose opposite extremes are $O = (0, 0)$ and $Q = (x_Q, y_Q)$ and the function $F_{R(0,Q)}(x, y) = 1$ if $(x, y) \in R(O, Q)$ and $x \neq x_Q$, $y \neq y_Q$, $F_{R(0,Q)}(x, y) = 0$ otherwise.

This bounded function is discontinuous but it is the limit of an increasing sequence of continuous functions $f_n(x, y)$, that assume the value 0 out of $R(O, Q)$ and 1 in a rectangle as $R(O, P)$ with P interior to $R(O, Q)$. Then the functional A can be prolonged to these new bounded functions obtaining a function φ defined in R:

$$\varphi(Q) = \varphi(x, y) = A(F_{R(0,Q)}) = \lim_n A(f_n).$$

Let $u(x, y)$ be such that $\Delta\Delta u = u(x, y) - u(x, O) - u(O, y) + u(O, O) = \varphi(x, y)$.
Consider a rectangular domain $R(P', P'')$, where $P' = (x', y')$, $P'' = (x'', y'')$, $x' < x''$, $y' < y''$, and the functions:

$F_{R(P',P'')} = 1$ for $(x, y) \in R(P',P'')$, $x \neq x''$, $y \neq y''$, $F_{R(P',P'')} = 0$ otherwise.
Let $Q' = (x', y'')$, $Q'' = (x'', y')$; it is

$$F_{R(P',P'')} = F_{R(0,P'')} - F_{R(0,Q')} - F_{R(0,Q'')} + F_{R(0,P')}$$

whence, since A is linear:

$$A(F_{R(P',P'')}) = \Delta\Delta_{R(P',P'')}\varphi = \Delta\Delta_{R(P',P'')}u.$$

Decompose now R in partial rectangular domains by means of the right lines of equation $x = x_i$, $y = y_j$ and define a function F that assumes the value $F_{i,j}$ in $R_{i,j}$ except in the edges of equation $x = x_{i+1}$ and $y = y_{j+1}$. It is $A(F) = \sum F_{i,j} A\,[F_{R(P_i,P_j)}] = \sum F_{i,j}\Delta_i\Delta_j\, u$, whence we deduce that the function u is of bounded variation, putting $F_{i,j} = 1$ for every i and j.
Moreover let $f(x, y) \in C(R)$: there exists a sequence of discontinuous functions of the previous type, $F_1, \ldots, F_n, \ldots$, that uniformly converges to f, then there exists finite the limit:

$$A(f) = \lim_n A(F_n) = \lim_n \sum F_{n_{i,j}}\Delta_i\Delta_j\, u = \iint_R f(x, y)d_x d_y u(x,y).$$

Caccioppoli then extends the procedure to the functions defined in a closed bounded set I, obtaining (*).

So, in this paper the author extends the linear bounded functional A, firstly defined in the class of the continuous functions, to the class of the bounded discontinuous functions that can be obtained as limits of continuous functions, in such a way that, as in the simpler case, it is:

$$|A(f)| \leq \text{Sup } |f| \cdot \text{tot.var. } u.$$

This is the starting point for a new general paper published in the same year on the C. R. Ac. Sc. Paris, where Caccioppoli proves that given a linear and bounded functional defined in the field of the continuous functions, as the Stieltjes integral or the Riemann integral, it is possible to extend it by continuity to a functional defined in all the set of the Baire functions, obtaining a constructive definition of the generalized integral of Stieltjes and also of Lebesgue [Caccioppoli 1926 b]. He anticipates in this way in a particular case the well-known Hahn- Banach theorem.

Applications of Lebesgue's Integration Theory and Daniell's Definition of Integral

Lebesgue's theory lends itself to a wide class of applications in an admirable manner, many of them being linked just to the developments and the generalizations of the idea of integral: in 1906, Fatou established in his thesis the famous theorem that the integral of the minimum limit of a sequence of measurable nonnegative functions is not greater than the minimum limit of the sequence of the integrals: this theorem joint with Lebesgue's theorem about the passage to the limit under the integral's sign (given for the first time by Lebesgue in his thesis in 1902 and proved in general in 1908) and with Beppo Levi's theorem also proved in 1906, gives a very powerful tool for the applications; in 1907 the theory enriches itself by the method of reduction of the multiple integrals introduced by Guido Fubini (Fubini 1907).

Moreover, in 1907 Riesz, who had took an interest in the Lebesgue's theory of integral after reading Lebesgue's monograph, proves that the square-summable functions on an interval (a, b) constitute a complete space of functions, today known as an Hilbert space (Riesz 1907); a result obtained in the same year but independently by E. Fisher (1875-1954).

The theorem of Fisher-Riesz being one of the great achievements of Lebesgue's theory, Fréchet determines immediately in 1907 the form of the functionals on this space and Riesz also obtains the same representation theorem: their fundamental papers about this topic are published on the same volume of the C. R. Acad. Sci. Soon other results of great interest for the integral equations theory follow: Riesz determines the representation of the linear and continuous functionals on the space of the continuous functions as we have already said and in 1910 establishes the expression of the linear and continuous functionals in the space L^p.

After the generalization of Lebesgue's theory of integration in 1915 given by Frèchet, our treatment could be considered concluded: but here we can consider as a last step of our way the definition of integral which Percy John Daniell (1889-1946) proposed in 1918 as a procedure very close to the functional methods used by Riesz and Young. His definition of integral does not need any initial development of measure theory and is so general that includes higher-dimensional spaces and Lebesgue-Stieltjes integral.

The starting point is a set T_o of numerical functions defined in an arbitrary set X. Daniell supposes T_o closed with respect to addition, multiplication for a constant, finite upper and lower bounds: in a first time the functions of T_o are supposed bounded. Then he considers a linear operator I on T_o such that

1) *if $f_1, \ldots, f_n, \ldots$ is a not increasing sequence of elements of T_o such that $\lim_n f_n(x) = 0$ for every x belonging to X then $\lim_n I(f_n) = 0$;*

2) *the operator I is isotone, that is if $f(x) \geq 0$ for every $x \in X$ then $I(f) \geq 0$.*

Such an operator is nowadays called a *Daniell measure*.

If for example T_0 coincides with the set of the Riemann integrable functions on an interval (a, b) then the application that associates every $f \in T_0$ with its Riemann integral extended to the interval (a, b) is a Daniell measure on T_0. But T_0 may be also the set of continuous functions on (a, b) and I the application that associates every $f \in T_0$ with its Stieltjes integral with respect to some not decreasing function or to some function of bounded variation.

Now the following question arises: is it possible to widen the set T_0 in a suitable way in order to prolong the definition of the operator I to a larger class of functions, preserving the fundamental properties of the integral and, if possible, improving the condition for the passage to the limit under the integral? To answer positively this question Daniell constructs the lower and upper integral, by an analogous method to the construction of the lower and upper Young's integrals and also in this case the integrals are defined for all the functions on X and therefore also for the unbounded ones which may assume infinite value.

Here is Daniell's construction: firstly, a set T_1, wider than T_0 is constructed and the operator I is extended to it. Precisely T_1 consists of those functions f such that there exists a not decreasing sequence of functions of class T_0, $\{f_n\}$, such that $f = lim_n f_n$.

Then $I(f_1) \leq I(f_2) \leq \cdots \leq I(f_n) \leq \ldots$ and therefore $lim_n I(f_n)$ exists, finite or infinite. Well, for every $f \in T_1$, Daniell puts $I(f) = lim_n I(f_n)$, where $\{f_n\}$ is whatever not decreasing sequence of elements of T_0, whose limit is f. It is possible to prove that $I(f)$ is well defined since its value is independent from the particular sequence that individuates f.

The function f is *summable* if f is of class T_1 and $I(f)$ is finite.

At this point Daniell defines upper and lower integrals of whatever function f. Precisely, given the function f , the upper integral, denoted by $\dot{I}(f)$ is by definition the greatest lower bound of $I(g)$ where g varies in the class T_1 in such a way that $g \geq f$. For this integral many properties analogous to the properties of the upper Riemann integral hold but also, differently from the upper Riemann integral, the passage to the limit under the integral sign for not decreasing sequence is allowed. Analogously, given the function f, the lower integral, denoted by $I.(f)$, is defined by $I.(f) = - \dot{I}(-f)$.

The function f is called *summable* if both upper and lower integrals are finite and coincide and their common value is the *integral* of f and denoted by $I(f)$.

Daniell establishes the fundamental theorems of the theory, that is the analogous of the Beppo Levi's theorem and the Lebesgue's Dominated Converge Theorem.

The formulation of the theory of integration by Daniell is very general, in line with the abstraction process that characterizes part of the mathematics in the twenties of the twentieth century. Then, in particular, we obtain Lebesgue's measure and integral in R^k considering as a functional the ordinary Riemann's integral. Observe that in such a case the procedure coincides exactly with Young's procedure and it is clear that the method here exposed is a generalization of it.

As in the case of Young's definition of integral, it is very simple to prove the Vitali-Charatheodory theorem, that is f is Lebesgue summable if and only if for every $\epsilon > 0$ there exist two functions g and h, g lower semicontinuous $g \geq f$, h upper semicontinuous $h \leq f$, such that $\int (g - h) dx < \epsilon$.

Indeed in this case, since T_1 consists of the lower semicontinuous functions, if f is summable then $I.(f) = \dot{I}(f)$, where:

$$\dot{I}(f) = \inf \{I(g), g \in T_1, g \geq f\} = \inf \{I(g), g \text{ lower semicontinuous}, g \geq f\}$$

and

$$I_{\cdot}(f) = -\dot{I}(-f) = -\inf\{ I(-h), -h \in T_1, -h \geq -f\}$$
$$= \sup\{-I(-h), h \text{ upper semicontinuous}, h \leq f\}$$
$$= \sup\{ I(h), h \text{ upper semicontinuous}, h \leq f\};$$

whence the thesis follows.

The Contribution of Italian Mathematicians in the Twenties of the Twentieth Century: Tonelli, Vitali and Caccioppoli

An interesting study to give a new version of Lebesgue theory of integral was made in Italy by Leonida Tonelli (1885-1946). In the first volume of [Tonelli 1921] this author intends to introduce Lebesgue's theory of sets in a completely new way, because he nearly wants to unveil the very nature of measurable sets, characterizing them from inside, by a construction from the bottom, closer to the constructive Borel's measure theory than to Lebesgue's one.

Here his try to give a good constructive counterpart of the definition of a measurable set and a Lebesgue integrable function is extensively reported to underline the interest this problem had in the thirties.

He introduces previously, for a subset *E* of the real line, the notion *to be equivalent to an interval* (also null, that is consisting of one point only) in the following way:
the set *E* is *equivalent* to an interval *I* if

1) *E is equal to I, that is it can be translated until it is placed upon I;*

2) *E is the union of a finite or countable number of disjoint intervals, such that if they are disposed consecutively, they form a limit interval equal to I;*

3) *E is the difference between an interval I' and a set equivalent, according to 1) and 2), to an interval I";*

4) *For every interval I' less than I, there exists a subset of E that is equivalent, according to 1), 2) and 3) to an interval greater than I'; and for every interval I" greater than I there exists a set containing E that is equivalent, according to 1), 2) and 3), to an interval less than I".*

Tonelli proves some properties of the sets equivalent to an interval, for example: a set *E* cannot be equivalent to two intervals of different lengths; two linear sets are called *equivalent* if they are equivalent to equal intervals; every linear closed set is equivalent to an interval; all the countable linear sets are equivalent to a null interval, a set E is equivalent to an interval if and only if it is Lebesgue measurable. Tonelli proves that the following characterization holds:

> *a linear set is equivalent to an interval if and only if the maximum limit of the lengths of the intervals equivalent to his closed components is finite and equal to the minimum limit of the lengths of the intervals equivalent to the sequences of not null and disjoint intervals which cover it* (Tonelli 1921, 118-119).

Since the previous condition holds even for Lebesgue measurable sets, Tonelli observes that a set is equivalent to an interval if and only if it is Lebesgue measurable. But he adds:

a Lebesgue measurable set is supposed to be bounded, on the contrary a set equivalent to an interval can be also unbounded (Tonelli 1921, 118-119 note 2).

From the characterization the following proposition follows:

A linear set E is equivalent to an interval if and only if for every natural number n there exist a closed component C of E and a sequence D of not null and disjoint intervals that covers E in such a way that the difference of the measures of the intervals equivalent to C and D is less than 1/n (Tonelli 1921, 122).

Now, using and modifying the previous characterization, Tonelli restricts the class of the measurable sets, in order to eliminate the sets generated by the use of the choice axiom. To this end he gives a definition of a constructive measurable set, he calls *pseudo-interval,* as:

a couple formed by a set E equivalent to an interval and a law which associates every natural number n with a closed set C_n enclosed in E and a sequence D_n of not null and disjoint intervals whose union covers E, such that the difference between the lengths of the intervals equivalent to D_n and C_n is less than 1/n. The measure of a pseudo-interval E, m(E), is the length of the interval equivalent to E (Tonelli 1921, 122).

The law in the previous definition is obviously supposed to be a constructive law, but Tonelli did not have yet the means to express it in a rigorous way. It is interesting to observe that in the thirties of the twentieth century the notion of computable function was introduced (but only for functions of integer numbers). This notion excluded the completely general functions that arise from the use of the choice axiom; almost surely Tonelli was influenced by the papers of Borel that in some way is a forerunner of the computable function theory. But realizing the difficulty of Borel's approach in defining constructive functions he explicitly requires that only operations that do not imply to turn to the choice axiom are admitted.

The class of the pseudo-intervals is closed with respect to the finite disjoint unions, the complements, the countable disjoint unions (but in this case Tonelli supposes that the sum of the series of the measures of the pseudo intervals is finite, since in his definition the sets can be boundless, but they must have finite measure.)

In the same way, again to eliminate functions generated by the use of the choice axiom, he gives the definition of almost-continuous function:

a function f(x) defined in the interval (a,b) is called almost-continuous if there exists a law that associates every natural number n with a closed subset E_n of (a,b) such that $m(E_n) > a\text{-}b - 1/n$ and f(x) is continous in E_n (Tonelli 1921, 131).

In this case also, the considered law must be thought as a constructive function.

Many properties of the almost-continuous functions are studied, particularly Tonelli observes that the *analytic* functions, that is "*the functions of x that can be constructed, with a given law, by means of a finite or countable number of additions, multiplications and passages to the limit*", are almost-continuous: for such functions he quotes (Lebesgue 1905).

As a matter of fact, Tonelli proves that all the almost-continuous functions he has defined constitute a subset of the set of the measurable Lebesgue functions, but he does not quote Vitali Lusin's theorem and does not prove the contrary implication since he clearly claims that in his construction "*the functions obtained by infinitely many arbitrary choices are not defined.*" Moreover, he adds:

> *until now we do not know functions defined in an interval (a,b) that are not quasi-continuous.*

As we have already observed, only some years after Robert Solovay proved that if the choice axiom is eliminated all the functions are Lebesgue measurable and therefore almost-continuous.

It would be possible also to believe that the measurable functions considered by Tonelli were all Borel measurable, but this is not true since few years after Lusin proved that there are simple measurable functions, obtained without the use of choice axiom, that are not Borel measurable (Lusin 1927).

The previous definitions are a try to obtain a constructive measure set theory. But this try was too complex, in fact some years after in a memoir Tonelli does not consider any more the equivalent sets to an interval but works only with sets that are *unions of a finite or countable number of intervals* (Tonelli 1923-24).

In the Introduction Tonelli claims:

> *This paper is an attempt to make easier, more elementary and therefore more acceptable by all, Lebesgue integral theory, by setting it free completely from measure theory.*

The new treatment, similar to those given by Young and Pierpoint, is proposed by defining Lebesgue's integral as an extension of Mengoli Cauchy integral, basing it on the notions of *union of a finite or countable number of intervals*, almost-continuous function and associated function with an almost-continuous function by a *union of a finite or countable number of open intervals D*.

Precisely:

> *A function f(x) defined on the interval (a, b), is called an almost continuous function in (a, b) if there exists a law that associates with every natural number n a union of a finite or countable number of open intervals* $D_n \subseteq (a, b)$ *such that its length is less than* $\frac{1}{n}$ *and f(x) is continuous in (a, b)-*D_n (Tonelli 1923, 110).

A *union of a finite or countable number of open intervals D* such that the function *f(x)* is continuous in (*a, b*)-D is said *associated with f(x)*.

Given an almost-continuous function *f(x)* on (*a, b*) and a *union of a finite or countable number of intervals D* associated with it, Tonelli calls *function associated with f(x) by D* and denotes by $f_D(x)$ the function equal to *f(x)* in (*a, b*)-*D* and such that in every interval of *D*, it varies linearly between the values it assumes in the extremes of the interval. So, every function associated with an almost-continuous function is continuous. Now the treatment is in some way like that of Young and the following theorem holds:

> *Given a bounded almost-continuous function f(x), consider a sequence of unions of a finite or countable number of intervals* D_n *enclosed in (a, b) such that the length of* D_n *is less than* $\frac{1}{n}$ *and such that f(x) is continuous in (a, b)-* D_n*, then the sequence of the associated functions* $f_{D_n}(x)$ *is a sequence of continuous functions converging to f(x) almost everywhere.*

Given the bounded almost-continuous function *f(x)* on the interval (*a, b*), Tonelli proves that:

> *the limit of the sequence of the Cauchy's integrals* $\int_a^b f_{D_n}(x)dx$ *of the functions* $f_{D_n}(x)$ *associated with f(x) when the lengths of the unions of a finite or countable number of open intervals* D_n *tend to zero exists and is finite* (Tonelli 1923-24, 117).

He defines this limit as the *integral of f(x) on* (*a, b*) and denotes it by the symbol $\int_a^b f(x)dx$; for unbounded functions he follows the method of de La Vallée Poussin.

Tonelli underlines the analogy with Riesz integral: but while Riesz approximates the integral of the given function by integrals of step functions, in his method the integral is approximated by integrals of continuous functions.

In a series of papers (Tonelli 1920) Tonelli deals with the problem of the research of the primitive functions; he briefly outlines the origins of the problem and observes that in the *Leçons* Lebesgue pointed out that every function which has a bounded derivative is the primitive of its integral and

$$f(x)-f(a)=\int_a^x f'(t)dt. \qquad (*)$$

If the function does not have a derivative, but a derivative number is bounded then (*) holds with the derivative number in place of *f'*(*x*). The equality (*) holds for every absolutely continuous function defined in accordance with Vitali, that is for every function *f*(*x*) defined in the interval [*a, b*] such that for every ε >0 it is possible to determine δ >0 such that for every finite set of intervals [a_1, b_1], [a_2, b_2], … [a_n, b_n] contained in [*a, b*] whose overall measure is less than δ, it is $\sum_{i=1}^{n} |f(a_i) - f(b_i)| < \varepsilon$.

Now it may be that a continuous function does have finite derivative in every point but is not absolutely continuous: Arnaud Denjoy solved this case by means of a method, he called *totalisation*, extending integration to a class of finite-valued not summable functions including all the functions with finite-valued derivatives. Then the problem was solved for every function with a finite derivative number.

Hahn and Beppo Levi gave examples of functions with a derivative number which is infinite on a perfect set and are not determined by it up to a constant (Biacino 2020, 30). Tonelli concludes this study proving the theorem:

> *A continuous function is determined up to a constant by one of its derivative numbers if and only if*:

1) *the derivative number is known almost everywhere*;

2) *neither of the two sets where the derivative number is* +∞ *or* - ∞ *is a perfect set*.

Besides Tonelli, some other Italian mathematicians devoted in the twenties their efforts to re-examine Lebesgue's integral and measure theory especially for a didactical purpose. For example, Vitali in 1929, in a lecture at Varsaw, gave the notion of integral for unbounded functions defined in unbounded sets (Vitali 1929). It is clear the didactical purpose of Vitali, that had been engaged until 1922 in the secondary school, with an active participation to the problems of the school teaching from 1908 until 1922. It is interesting how he reproposes in part, as we will see, the Riesz method revisited and exposed in a simplified manner without neglecting measure theory. As a matter of fact, since 1903 Vitali had developed, independently from Lebesgue, his own measure theory, equivalent to Lebesgue's one, proving a series of results by original methods.

The measure of a set is defined by him as the exterior Lebesgue measure, once established that the set is measurable in the following way. Vitali calls a *simple Borel set* every set that is the finite or denumerable union of intervals, bounded or not, pairwise without interior points in common. He calls *cover* of a subset *a* of a right line *r* every simple Borel set such that the given subset is enclosed in it. He calls *extension* of *a* the greatest lower bound of the lengths of the covers of *a*. Given a set *a* and a cover C of *a*, the points that belong to some interval of C but do not belong to *a* form a set *a'*. The greatest lower bound of the extensions of the sets *a'* corresponding to the covers of *a* is called *anomaly of a*. A set is said measurable if its anomaly is zero and in this case its extension is called its *measure*. Then Vitali gives some preliminary definitions to determine under what conditions a not bounded function in a not bounded measurable set is *summable*.

A function $f(t)$ defined almost everywhere in a measurable set G is called *almost-constant* if $f(t)$ assumes only a finite or countable number of different and finite values and if every set on which it assumes the same value is measurable.

If l_n ($n = 1, 2, \ldots$) are the different values $f(t)$ assumes and G_n is the set where it assumes the value l_n, then $f(t)$ is called *summable* if the series $\sum l_n m(G_n)$ is with finite terms and is *absolutely convergent* (with the condition $0 \cdot \infty = 0$). In such a case the sum of the series is the *integral* of the function $f(t)$ extended to G.

If $f(t)$ is a measurable (in the usual way) function in G then every almost–constant function $F(t)$ such that $f(t) \leq F(t)$ almost everywhere in G is called a *majorant of* $f(t)$; and a *minorant* of $f(t)$ is an almost-constant function $H(t)$ such that $H(t) \leq f(t)$ almost everywhere in G.

A measurable function defined in the measurable set G is called *summable* if it admits a summable majorant function and a summable minorant function in G. It is simple to prove that this happens if and only if the function is Lebesgue summable. In such a case the integrals of its summable majorants and of its summable minorants form two contiguous classes of real numbers, whose element of separation is called *integral of* $f(t)$ *extended to* G. It is possible to prove that this new integral coincides with the Lebesgue integral.

We conclude this section exposing the original contribution to the definition of Lebesgue's integral given by the young Renato Caccioppoli (1904-1959) who represents the ideas of the Italian school of analysis of those years, dominated by the teacher of Caccioppoli, Mauro Picone (1885- 1977) and by L. Tonelli and animated also by the ambition for a role in the international competition.

In 1928 Caccioppoli published an important memoire devoted to the integration of discontinuous functions (Caccioppoli 1928a), very close to the ideas of Borel, Riesz, Tonelli, Young, Daniell, but his main problem is not only to make his theory independent from the Lebesgue measure theory as it has been for many other authors, but, in the same order of ideas of (Caccioppoli 1926b), what is essential for the Author is to produce a theory that starts with the Cauchy integral considered as a linear and continuous operator on the field of the continuous functions and gives a prolongation of it by means of a natural extension to a wider field of functions, every one of them being continuous except in a set of arbitrarily small measure.

In (Caccioppoli 1928a) the following program is exposed: to search a definition built on an abstract basis in such a way that it can be applied both to multiple integrals and to the Stieltjes integral; moreover, the fundamental theorems about the passage to the limit under the integral sign must be preserved and improved. The Author carries out his program resorting to a general theory of the limits, introduced few years before, in 1923, by M. Picone[14], in this way: he first considers whatever function *F* defined and bounded in a bounded rectangle *R*. For every closed subset *I* of *R*, where F is continuous, he considers a function f_I that is continuous in R and coincides with F in *I* and suppose that there exists a constant *M* such that f_I is not greater than *M* for every *I*. Consider the values $\int_R f_I dT$, they

[14] The notion of ordered variable or net had been introduced almost contemporarily by E. H. Moore and L. H. Smith in 1922 in: *A General Theory of Limits*, on the Amer. Journ. of Math.

constitute a bounded set. Caccioppoli applies at this point the theory of the ordered variables exposed in the first chapter of (Picone 1923).

The maximum and minimum limit of the ordered variable so defined are finite: if they coincide their common value is called the *integral* of F on R and is denoted by $\int_R F(P)dT$ or by $\int_R F\, dT$ and F is said *integrable*.

Caccioppoli introduces at this point an additive function on the open subsets of R in this way: let E be an open set and put:

$$\varphi(E) = \sup\{|\int_R F\, dT|: F \text{ continuous in } R,\ F(x)=0 \text{ for } x \in R\text{-}E,\ |F(x)| \leq 1\}.$$

He proves the following fundamental theorem:

The bounded function F is integrable if and only if for every $\varepsilon>0$ there exists a closed set I such that F is continuous in I and $\varphi(R\text{-}I) < \varepsilon$.

If E is a bounded set the integral of whatever function F defined on E is given in the usual way considering a bounded interval R that contains the set E and introducing the function F_E' coinciding with F on E and with 0 in R-E. Then F is *integrable* on E if and only if F_E' is integrable on R and the *integral* $\int_E F(P)dT$ is by definition given by $\int_R F_E'\, dT$.

Caccioppoli proves that the sets whose characteristic function is integrable constitute a family which contains all the Borel sets and the integral of the characteristic functions that are integrable is an additive measure that coincides with the Lebesgue measure. The set function previously defined on the open sets, $\varphi(E)$, coincides with the integral of the characteristic function of the set E. Moreover, every integrable function on a measurable set is also Lebesgue measurable. It follows that the so defined integral coincides with the Lebesgue integral, the Stieltjes integral can be analogously defined. Then, by the previous characterization, it follows that a bounded function is integrable if and only if it is, in accordance with Tonelli's definition, *almost continuous*.

Caccioppoli proves in general, that is for functions of more variables, the Severini-Egoroff Theorem about the almost continuity of the sum of a converging series of continuous functions. For the proof he uses a noteworthy *Generalized Arzela's Lemma*, established by him, that with modern words can be exposed in the following way:

Given an abstract space X, a σ – algebra F of elements of X and a countably additive measure φ on F, consider a sequence $I_1, I_2, \ldots, I_n, \ldots$ of elements of F: if there exists $\sigma > 0$ such that for every $n \in N$ it is $\varphi(I_n) > \sigma$ then there exist points which belong to infinitely many I_n.

A series of theorems about the passage to the limit under the integral sign follows. The first is of course a new form of the Lebesgue integral theorem for the limit function of a sequence of bounded integrable functions:

If a sequence of integrable functions: $f_1, f_2, \ldots, f_n, \ldots$ converges almost everywhere in E to the bounded function f, then f is integrable on E; and if the functions are uniformly bounded then it is possible to pass to the limit under the integral sign.

Because of this theorem, a limit of a sequence of integrable functions is a measurable function.

Caccioppoli adds to the previous theorem the following proposition:

If the function F is such that, given $\varepsilon > 0$, it is possible to find two integrable functions on E, h_ε and k_ε, such that

$$|F-h_\varepsilon| < k_\varepsilon \text{ on E} \quad \text{and } \int_E k_\varepsilon dT < \varepsilon,$$

then F is integrable on E.

For the proof consider an infinitesimal sequence $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots$ such that the series $\sum \varepsilon_n$ is convergent and let $\sigma > 0$ and H_n be the set of the points of E where $k_{\varepsilon_n} > \sigma$. Now it is:

$$\varepsilon_n > \int_E k_{\varepsilon_n} dT > \sigma \int_{H_n} dT.$$

Therefore:

$$|F - h_{\varepsilon_n}| < k_{\varepsilon_n} < \sigma \text{ in } E-H_n \quad \text{with } \varphi(H_n) < \frac{\varepsilon_n}{\sigma}.$$

Given $\varepsilon > 0$ there exists a natural m such that $\sum_{n\geq m}^{\infty} \varepsilon_n < \varepsilon\sigma$ and therefore for every $n \geq m$ it is

$$|F - h_{\varepsilon_n}| < \sigma \text{ in } E - \cup_{n\geq m} H_n, \text{ with } \varphi(\cup_{n\geq m} H_n) < \varepsilon.$$

Therefore $|F - \lim h_{\varepsilon_n}| < \sigma$ in $E - H$ with $\varphi(H) = 0$; since σ is arbitrary we conclude that $F(x) = \lim h_{\varepsilon_n}$ and by the previous theorem F is integrable.

Finally, Caccioppoli proves the following extension and completion of the last theorem:

Given a function F, a number $\mu > 0$ and two sequences of functions integrable on E, $\{h_n\}_{n\in N}$ and $\{k_n\}_{n\in N}$, such that

$$|F - h_n|^\mu < k_n \text{ on E} \quad \text{and } \lim_n \int_E k_n dT = 0,$$

then F is integrable and if $\mu \geq 1$ then $\lim_n \int_E h_n dT = \int_E F dT$.

Caccioppoli defines also the integral of an unbounded function. Let F be a function defined in a rectangular domain R, suppose the function F almost continuous and constantly nonnegative and consider the ordered variable defined by $\int_R f_I \, dT$, where for every $\varepsilon > 0$ there exists a closed set I such that $\varphi(R - I) < \varepsilon$, f_I is continuous in R and coincides with F in I. The ordered variable in this case is not bounded and therefore his maximum limit is $+\infty$.

If its minimum limit is finite, it is called the *integral*, is denoted by $\int_R F \, dT$ and the function F is called *summable*.

In every case a function F is called *summable* if the nonnegative functions $|F|$ and $|F| - F$ are summable and the integral is defined by $\int_R F \, dT = \int_R |F| \, dT - \int_R (|F| - F) \, dT$.

Caccioppoli observes that his construction gives an equivalent definition to that of de La Vallée Poussin; he proves that:

if F is nonnegative and summable then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every subset I of R such that $\varphi(R - I) < \delta$ it is $\int_R F \, dT - \int_R f_I \, dT < \varepsilon$.

The previous property is then extended to a family of functions by the following new notion.

The functions of a family of nonnegative summable functions F are called *uniformly summable* if for every $\varepsilon > 0$ there exists $\sigma > 0$ such that if $\varphi(R\text{-}I) < \sigma$ then $\int_R F\,dT - \int_R f_I\,dT < \varepsilon$ for every function F of the family. Summable functions of a family are also called *uniformly summable* if their absolute values are uniformly summable.

The following theorem holds:

> *If the functions of the almost everywhere convergent sequence $(F_n)_n$ are uniformly summable, then the limit function F is summable too and $\int_R F dT = \lim_n \int_R F_n\, dT$.*

Caccioppoli in general proves that the integral function of a summable function $\int_E F dT$ as a function of the set E is *absolutely continuous* and writes the following proposition:

> *Let F be a summable function: then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every set E if $\varphi(E) < \delta$ then $\int_E |F| dT < \varepsilon$.*

A sufficient condition in order that the summable functions of a family are uniformly summable is that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every set E if $\varphi(E) < \delta$ then $\int_E |F| dT < \varepsilon$ for every function F of the family[15]. In particular, this condition is satisfied if there exists a nonnegative summable function such that the absolute values of all the functions of the family are not greater than it, that is the Lebesgue's theorem holds.

> *Given the sequence F_n of summable functions tending to a function F, if there exists a nonnegative summable function G such that $F_n \leq G$ for every natural n, then F is summable and $\int_E F dT = \lim_n \int_E F_n\, dT$.*

Among many other things Caccioppoli observes that:

> *The functions of a family of uniformly summable functions have uniformly absolutely continuous integrals* (Caccioppoli 1928 a, 26).

Then the functions of a family are uniformly summable if and only if they have uniformly absolutely continuous integrals.

If we compare Tonelli's and Caccioppoli's approaches to integration it is evident that they are equivalent, but the second is much more abstract; Tonelli considers preferably functions of one variable on which he operates almost physically, see for example the way he prolongs an almost continuous function in a continuous function, while Caccioppoli uses more general theorems to prolong for example a continuous function in a closed bounded set enclosed in a rectangle R into a continuous function on R. Moreover, Tonelli starts with the definition of integral for an almost continuous function, as Lebesgue starts equivalently with measurable functions, whilst in the construction of Caccioppoli the definition of integral

[15] The integrals of a family of functions for which the previous condition holds are called uniformly absolutely continuous in (Vitali 1907), where it is proved that a sufficient condition in order to pass to the limit under the integral sign for a sequence of converging functions defined in a measurable subset of the real number set R whose measure is finite is that their integrals are uniformly absolutely continuous. The condition is also necessary if it is possible to pass to the limit under the integral extended to every subset of the given one. The first definition of an absolutely continuous real function was given by (Vitali 1904-05). It is formulated again for a function of two variables in (Vitali 1908). The notion is deeply studied in 1910 by Lebesgue: the central part of (Lebesgue 1910) indeed is a re-examination of the definitions and the proofs of the Vitali's widely quoted paper of 1908: real additive set functions are considered instead of integral functions, for them a notion of derivative is defined, in general.

is independent from this preliminary hypothesis, which is obtained after as an equivalent condition in order a function to be integrable.

In the same year in (Caccioppoli, 1928 b) the Author introduces a definition of Stieltjes integral[16] with respect to a function of bounded variation *u* for a measurable and not necessarily bounded function *F*[17], also in this case, as for the Lebesgue integral (Caccioppoli 1928a), in application of the theory of the limits of the ordered variables in the following way: given $\varepsilon>0$ let *I* be a closed set such that the summable function *F*, defined in the rectangle *R*, is continuous in *I* and $\varphi(R\text{-}I) < \varepsilon$ and let f_I be a continuous function in R that coincides with F in *I*. Let us suppose *u* nonnegative, *F* and f_I nonnegative and order the Stieltjes integrals $\int_R f_I(P)du$ as the intervals *I*. Then the maximum limit of the ordered variable $\int_R f_I(P)du$ is infinite; if the minimum limit is finite then the function *F* is called *summable* on *R* with respect to *u* and the minimum limit is called the *Lebesgue Stieltjes integral* of F on R. One can proceed in a canonical way if the functions F and *u* are of variable sign.

Given a family of functions F defined in the interval R and summable with respect to a function of bounded variation *u*, they are called *uniformly summable with respect to u* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varphi(R\text{-}I) < \delta$ then $|\int_R |F(P)|du - \int_R f_I(P)du| < \varepsilon$, for every function F of the family, f_I being a continuous function in *I* that coincides with |*F*| in *I*.

The previous condition allows a partial generalization of Vitali's Theorem about the passage to the limit under the integral sign:

If the functions $F_1(P), F_2(P), \ldots$ are uniformly summable with respect to u and F(P) = $\lim_n F_n(P)$ then F(P) is summable with respect to u and $\int F(P)\,du = \lim_n \int F_n(P)\,du$.

The next step done by the Author is to avoid in the previous condition the dependence from the function u. He succeeds in this program introducing the notion of *family of functions of uniformly bounded variation*. Indeed, the functions of a sequence $\varphi_1(P), \varphi_2(P), \ldots$ are called of *uniformly bounded variation* if the limit relation $\lim_n V(\varphi_m, E_n) = 0$, where $V(\varphi_m, E_n)$ is the variation of the set function φ_m in the set E_n, is uniform with respect to *m* for every sequence of pairwise disjoint sets $\{E_n\}$[18].

Caccioppoli proves that the integrals $\int F_1(P)\,du, \int F_2(P)\,du, \ldots$ are of uniformly bounded variation if and only if they are uniformly absolutely continuous with respect to u, that is if $\varphi(E)$ tends to zero then $\int_E F_n(P)du$ tends to zero uniformly with respect to n.

It is now possible to prove the Generalized Vitali's Theorem:

[16] A study on the Stieltjes integral in the most general conditions is present in the last paper published by Caccioppoli in 1955. In it the Author considers the derivative, or the minimum and maximum derivative, of a function *F(x)* of a real variable with respect to another function *g(x)* given by $\lim_{h\to 0}\frac{F(x+h)-F(x)}{g(x+h)-g(x)}$ or by the minimum or maximum limit of the previous ratio. In order to determine a primitive of *F(x)*, given *g(x)*, Petrovski and Caccioppoli in the same year 1934 have proved that, if *g(x)* is continuous, a function that is derivable with respect to *g(x)* is determined by its derivative on an interval up to a constant. In 1955, among other things, Caccioppoli proves that, for the Lebesgue-Stieltjes integral $\int Fdg$, if g(x) is continuous but not of bounded variation, a primitive exists if and only if the integrand is summable and every integral function is a generalized primitive, that is it admits the integrand function as a derivative almost everywhere, in a sense specified by the Author. The theory is dominated by the hypothesis of the absolute continuity of *F(x)* with respect to *g(x)*, case in which every primitive coincides with the integral function up to a constant (Caccioppoli 1955).

[17] That is such that *for every $\varepsilon>0$ there exists a closed set I such that F is continuous in I and $\varphi(R\text{-}I) < \varepsilon$*, as in the definition of almost continuous function given by Tonelli.

[18] Obviously all the considered sets are supposed Borel measurable. The variation of a set function g in a set E is defined as the least upper bound of the sums $\varphi(E_1) + \varphi(E_2) + \ldots + \in(E_n)$ where $E_1, E_2, \ldots E_n$ varies in the class of the partitions of *E*.

If F(P) = $\lim_n F_n(P)$ and if the integrals of the functions $F_1(P), F_2(P), \ldots$ are of uniformly bounded variation then $\int F(P)\,du = \lim_n \int F_n(P)\,du$.

Notice that the condition about the indefinite integrals to be of uniformly bounded variation is more flexible than the condition to be uniformly absolutely continuous with respect to *u* since it is independent from the function *u*: indeed, it suggests a more general theorem about the passage to the limit under the integral sign Caccioppoli expresses in the following way:

If F(P) = $\lim_n F_n(P)$, if $\lim_n u_n(P) = u(P)$ and if the indefinite integrals, $\int F_1(P)\,du_1$, $\int F_2(P)\,du_2, \ldots$ are of uniformly bounded variation then $\int F(P)\,du = \lim_n \int F_n(P)\,du_n$.

The notion of a family of *functions of uniformly bounded variation* will be found independently in 1947 by the mathematician V. M. Dubrowskii[19] who considers *families of uniformly additive set functions*, the name used today. Some results of J. Diedonné[20] about the convergence of sequences of measures are in the same order of ideas. And a deep study of the uniformly additive families of measures is given in the Chapt. 5 of (Cafiero 1959).

Note that in the previous exposition of the Lebesgue Stieltjes integral the Author still refers to functions of bounded variation as in Vitali's definition, that is as point functions and not set functions, while in other papers of the same year (he wrote fourteen papers in 1928) he considers generally set functions of bounded variation. This is a little difficulty when compared with the richness of new ideas and results in his work of that period.

Conclusion

We conclude here the treatment of the many presentations of the fundamental theory of the Lebesgue integral after Lebesgue: they were proposed by many mathematicians of many countries of Europe and USA, French, English, Belgian, Austrian, German, Russian, Polish, Italian, Hungarian, United States Authors were deeply interested in the new theory and thought to build other versions of it, sometimes exclusively for didactical purposes, some time giving generalizations and new correlate notions that would contribute to create the completely general character of the mathematics of today. We have investigated the various aspects of their theories, and we can now observe in how many ways they have solved a so delicate and unavoidable problem. More than the technical aspects, the inventiveness and the fantasy of the various Authors, that however are not underestimated, and the important didactical purpose just evoked, we have pointed out to the philosophy below their formulations: almost all are animated by a modern constructive idea to make mathematics; but someone as well is driven by the also modern spirit of the time that is the idea of abstractionism which in mathematics becomes functionalism. A purpose of this exposition is to show that they wanted, besides opening new areas of research, to link the new ideas to the use of methods that took root in their cultural own tradition, with the prospects to perpetuate and to hand down it to posterity.

[19] V. M. Dubrowskii, 1947, Matematiceskij Sbornik (N.S.), 20, 317-19 e V. M. Dubrowskii, 1947, Doklady Akademii Nauk SSSR,(N.S.) 58, 737-40.

[20] J. Diedonné,1951. *Sur la convergence the suites de measures the Radon*, Anais Acad. Brasil.Ci.23, 21-38, 277-282.

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